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The multiscale expansions of difference equations in the small lattice spacing regime, and a vicinity and integrability test: I

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Abstract

We propose an algorithmic procedure (i) to study the 'distance' between an integrable PDE and any discretization of it, in the small lattice spacing ϵ regime, and, at the same time, (ii) to test the (asymptotic) integrability properties of such discretization. This method should provide, in particular, useful and concrete information on how good is any numerical scheme used to integrate a given integrable PDE. The procedure, illustrated on a fairly general ten-parameter family of discretizations of the nonlinear Schrödinger equation, consists of the following three steps: (i) the construction of the continuous multiscale expansion of a generic solution of the discrete system at all orders in ϵ, following Degasperis *et al* (1997 *Physica D* **100** 187–211); (ii) the application, to such an expansion, of the Degasperis–Procesi (DP) integrability test (Degasperis A and Procesi M 1999 Asymptotic integrability *Symmetry and Perturbation Theory*, SPT98, ed A Degasperis and G Gaeta (Singapore: World Scientific) pp 23–37; Degasperis A 2001 Multiscale expansion and integrability of dispersive wave equations Lectures given at the *Euro Summer School: 'What is integrability?' (Isaac Newton Institute, Cambridge, UK, 13–24 August)*; *Integrability* (*Lecture Notes in Physics* vol 767) ed A Mikhailov (Berlin: Springer)), to test the asymptotic integrability properties of the discrete system and its 'distance' from its continuous limit; (iii) the use of the main output of the DP test to construct infinitely many approximate symmetries and constants of motion of the discrete system, through novel and simple formulas.

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1. Introduction

Given a partial differential equation (PDE) and a partial difference equation (PΔE) discretizing it, it is interesting to know, when the lattice spacing ϵ is small, 'how close' the two models

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are. In particular, if the PDE is integrable, it is important to have a way to establish if such a discretization preserves integrability or, at least, how ‘close’ it is to an integrable system, detecting the order, in ϵ, at which the discretization departs from integrability and, correspondingly, the time scale at which one should expect numerical evidence of nonintegrability and/or chaos. In addition, given a PDE and two PΔEs discretizing it, it is also interesting to know, when the lattice spacing ϵ is small, ‘how close’ the two PΔEs are.

In this paper we propose to answer these basic questions in the following way. Concentrating on an integrable PDE and on a PΔE discretizing it,

(1) we construct and study in detail the multiscale expansion at all orders of a *generic long-wave solution* of the PΔE under scrutiny, generated in the small ϵ regime, following the procedure developed in [1]. At $O(1)$, the leading term u of such an asymptotic expansion satisfies the integrable PDE; to keep the expansion asymptotic, we eliminate the secularities due to the linear part of the PΔE, arising at each order, introducing infinitely many slow (time) variables and establishing that the evolution of u with respect to such slow times is described by the infinite hierarchy of commuting flows of the integrable PDE, as in [1];
(2) we make use of the asymptotic integrability test developed by Degasperis–Procesi (DP) in [2, 3] on such a multiscale expansion to test, at all orders, the ‘asymptotic’ integrability properties of the PΔE; in particular, detecting the order in ϵ (and, correspondingly, the time scale) at which the discretization departs from integrability. At this time scale, for instance, numerical simulations are expected to give some evidence of nonintegrable and/or chaotic behavior;
(3) we finally show how to make use of the main output of the DP test to construct infinitely many ‘approximate’ symmetries, at a required order in ϵ, of the PΔE under scrutiny, using novel and simple formulas.

Recent studies on the performances, as numerical schemes for their continuous limits, of PΔEs possessing the same (continuous) Lie point symmetries as their continuous limits can be found in [4–8]. Studies on the performances, as numerical schemes for their continuous limits, of integrable discretizations of integrable PDEs can be found, for instance, in [9] and [10]; in this case, the integrable discretization possesses infinitely many exact generalized symmetries and constants of motion in involution at any order in ϵ, reducing to the generalized symmetries and constants of motion of the integrable PDE in the continuous limit. The PΔEs selected by our approach possesses infinitely instead many approximate generalized symmetries and constants of motion in involution at the required order in ϵ (see section 3.1), reducing to the generalized symmetries and constants of motion of the integrable PDE in the continuous limit.

The procedure we propose should allow one to have a control on the ‘distance’ between the PΔE and its continuous limit, as well as the distance between two different discretizations of the same PDE. Indeed, suppose we construct an asymptotic expansion of the form $\psi = u + O(\epsilon^\alpha), \alpha > 0$, where ψ is a generic long-wave solution of the PΔE and u is the corresponding solution of its continuous limit; if, at $O(\epsilon^\beta), \beta > 0$, the P$\Delta$E passes the DP test, we infer that $||\psi - u|| = O(\epsilon^\alpha)$ at time scales of $O(\epsilon^{-\beta})$, where $||\cdot||$ is the uniform norm w.r.t x and t (the norm used to test the asymptotic character of the generated multiscale expansion). In this way, since we control the distance between ‘generic long-wave solutions’ of the PΔE and of its continuous limit, we also control the distance between the PΔE and its continuous limit. In addition, if the multiscale expansions of two different discretizations of the same PDE pass the DP test at $O(\epsilon^\beta)$, we infer, from the triangular inequality, that $||\psi - \phi|| < ||\psi - u|| + ||\phi - u|| = O(\epsilon^\alpha)$ at time scales of $O(\epsilon^{-\beta})$, where ψ, ϕ are long-wave solutions of the two different discretizations of the PDE corresponding to the same initial-boundary data; therefore, we have a control also on the distance between the two different discretizations of the same PDE.

Some historical remarks are important, at this point, on the theory of multiscale expansions in connection with integrable systems to put the results of this paper into a proper perspective. Multiscale expansions of a given PDE are very useful tools for investigating the properties of such a PDE and for identifying important model (universal) equations of physical phenomena. For instance, if the original nonlinear PDE has a dispersive linear part, a small amplitude quasi-monochromatic wave evolving according to it develops a slow spacetime amplitude modulation described by the celebrated nonlinear Schrödinger (NLS) equation [11–15] (see also [16–18])

$$\mathrm{i}u_t + u_{xx} + 2c|u|^2u = 0, \qquad u = u(x,t) \in \mathbb{C}, \tag{1}$$

integrable if c is a real constant [19]. Considering, instead, three monochromatic waves and imposing a suitable resonance condition on their wave numbers and dispersion relations, one generates another integrable universal model, the three-wave resonant system [20]. In the above two examples, the expansion is constructed around 'approximate' particular solutions of the original PDE (the monochromatic waves). It is also possible to expand around the 'exact' particular solutions of the original PDE; for instance, as shown in [21], expanding around the exact solution $u_0 = \exp(2\mathrm{i}ct)$ of (1), the first nontrivial term of the asymptotic expansion evolves according to another important model equation: the Korteweg–de Vries (KdV) equation [22], sharing with NLS the property of integrability [23]. Since multiscale expansions preserve integrability [21], (i) if the original PDE is a 'C-integrable' system (i.e. it is linearized by a 'change of variables' [18, 24], like the Burgers equations [25]), the model equation generated by it is linear [18, 24]; (ii) if the original PDE is an 'S-integrable' system, or soliton equation (like the NLS equation), integrated in a more complicated way via a Riemann–Hilbert or $\bar{\partial}$-problem [26–29], the model equation generated by it is also 'S-integrable'; and vice versa, (iii) if the model equation generated by the expansion is not integrable, then the original equation is not integrable too (indeed, if it were integrable, the integrability preserving multiscale expansion would generate an integrable model equation). This criterion has been used in [18, 30–32] as a simple test of integrability. In addition, the universal character of the identified model equations (NLS, KdV or others) is also the reason why model equations possess very distinguished mathematical properties and, often, they are integrable [18, 30, 24].

Multiscale expansions can also be carried, in principle, to all orders and, as a consequence of eliminating the secular terms at each order, a sequence of slow time variables $t_n = \epsilon^n t$ must be introduced and the dependence of the leading term of the expansion on such slow times is described by the hierarchy of commuting flows of the integrable model equation [1]. This multiscale expansion at all orders has been used in [2, 3] to build an efficient asymptotic integrability test for the original PDE (see section 3 for more details on such a test). An alternative asymptotic integrability test, based on the existence of approximate symmetries for the original PDE, can be found in [33]. The ideas and procedures developed in [1–3] have been recently used to build an integrability test also for PΔEs [34–36]; in this approach, one expands, as for the PDE case, around the approximate or exact particular solutions of the PΔE under investigation, obtaining a continuous multiscale expansion at all orders, following [1], and applying on it the DP test. The main difference between the procedure followed in [34–36] and the results of this paper is the following. The standard multiscale approach used in [34–36], obtained expanding around approximate or exact particular solutions of the PΔE under investigation, cannot give information on how close this PΔE and its continuous limit are, the main goal of the present paper. The common features of the procedure in [34–36] and of that used in this paper are that, in both cases, one constructs, from the given PΔE, continuous multiscale expansions carried to all orders, as in [1], and one applies to them the DP integrability test. Therefore, both procedures can be used to test the integrability and the asymptotic integrability of the original PΔE. A deeper comparison of the effectiveness of these two procedures to test the integrability of a given PΔE is postponed to a subsequent paper.

Another integrability test for PΔEs is the so-called symmetry approach [37], based on the existence of higher order symmetries and originally developed to test the integrability of PDEs [38, 39].

The results of this paper are illustrated on the basic prototype example of the NLS equation (1), starting from the following discretization of it:

$$\begin{aligned}&\mathrm{i}\psi_{n,t}+\epsilon^{-2}(\psi_{n+1}+\psi_{n-1}-2\psi_n)+F(\psi_{n-1},\psi_n,\psi_{n+1})=0,\\&F(\psi_{n-1},\psi_n,\psi_{n+1}):=2a_1|\psi_n|^2\psi_n+a_2|\psi_n|^2(\psi_{n+1}+\psi_{n-1})\\&\quad+a_3\psi_n^2(\bar\psi_{n+1}+\bar\psi_{n-1})+a_4\psi_n(|\psi_{n+1}|^2+|\psi_{n-1}|^2)\\&\quad+a_5\psi_n(\bar\psi_{n+1}\psi_{n-1}+\psi_{n+1}\bar\psi_{n-1})+a_6\bar\psi_n(\psi_{n+1}^2+\psi_{n-1}^2)\\&\quad+2a_7\bar\psi_n\psi_{n+1}\psi_{n-1}+a_8(|\psi_{n+1}|^2\psi_{n+1}+|\psi_{n-1}|^2\psi_{n-1})\\&\quad+a_9(\psi_{n+1}^2\bar\psi_{n-1}+\psi_{n-1}^2\bar\psi_{n+1})+a_{10}(|\psi_{n+1}|^2\psi_{n-1}+|\psi_{n-1}|^2\psi_{n+1}),\end{aligned}\tag{2}$$

where the constant coefficients a_j, $j=1,\ldots,10$, are real, reducing to (1) in the natural continuous limit in which the lattice spacing $\epsilon\to 0$ and $n\epsilon\to x\in\mathbb{R}$, $\psi_n(t)\to u(x,t)$, with

$$c=\sum_{j=1}^{10}a_j.\tag{3}$$

The ten-parameter family of equations (2) has recently been taken in [40] as the starting point of an analysis devoted to the identification of discretizations of NLS that possess, at the same time, a solitary wave and a breather solution reducing, respectively, to the one soliton and breather solutions of the NLS equation (1), in the continuous limit $\epsilon\to 0$. We remark that, rescaling the dependent variable, one can always introduce one normalization for the ten coefficients; or for instance, choose one of these coefficients, say a_j, to be $\mathrm{sign}(a_j)$ or, better for our purposes, normalize the sum (3) of the ten coefficients to coincide with the prescribed coefficient c of the NLS equation (1).

The linear part of the discrete NLS (dNLS) (2) is the standard discretization of $(\mathrm{i}u_t+u_{xx})$; its nonlinear part is uniquely fixed by the following, physically sound, properties [40]. (a) Equation (2) must possess the gauge symmetry of first kind (i.e. if ψ_n is a solution, $\psi_n\mathrm{e}^{-\mathrm{i}\theta}$ is a solution too, where θ is an arbitrary real parameter), corresponding to the infinitesimal gauge symmetry $-\mathrm{i}\psi_n$. (b) The nonlinearity is cubic; i.e. it is the weakest nonlinearity compatible with the above gauge symmetry. (c) Only the first neighbor interactions are considered. (d) Equation (2) is invariant under the symmetry transformation $\psi_{n\pm1}\to\psi_{n\mp1}$ (space isotropy).

The dNLS (2) contains, in particular,

(1) the integrable Ablowitz–Ladik (AL) equation [41]

$$\mathrm{i}\psi_{n,t}+\epsilon^{-2}(\psi_{n+1}+\psi_{n-1}-2\psi_n)+a_2|\psi_n|^2(\psi_{n+1}+\psi_{n-1})=0,\tag{4}$$

for $a_j=a_2\delta_{j2}$, $j=1,\ldots,10$;

(2) the discretization

$$\mathrm{i}\psi_{n,t}+\epsilon^{-2}(\psi_{n+1}+\psi_{n-1}-2\psi_n)+2a_1|\psi_n|^2\psi_n=0,\tag{5}$$

for $a_j = a_1\delta_{j1}, j = 1, \ldots, 10$, relevant in several applications [42–46] whose nonintegrability has been recently shown in [34, 35] using the DP test;

(3) the discretization corresponding to

$$a_{10} = a_8, \qquad a_1 = a_4 = a_5 = a_6 = a_7 = a_9 = 0, \tag{6}$$

with a_2, a_3, a_8 arbitrary, possessing a solitary wave as well as a breather solution reducing, respectively, to the one soliton and breather solutions of the NLS equation in the limit $\epsilon \to 0$ [40];

(4) the discretization corresponding to

$$a_8 = a_3, \qquad a_2 = 2a_3, \qquad a_4 = 2a_6, \qquad a_5 = a_7 = a_9 = a_{10} = 0, \tag{7}$$

where a_1, a_3, a_6 are given in terms of physical quantities, describing coupled optical waveguides embedded in a material with Kerr nonlinearities [47];

(5) the discretization corresponding to

$$a_4 = a_2, \qquad a_1 = a_3 = a_6 = a_8 = a_2/2, \qquad a_5 = a_7 = a_9 = a_{10} = 0 \tag{8}$$

(a particular case of (7)), appearing in the modeling of the Fermi–Pasta–Ulam problem [48].

For special values of the coefficients a_j the dNLS equation (2) is Hamiltonian. For instance, equations (4), (5), (7) and (8) are Hamiltonian [47].

If $0 < \epsilon \ll 1$, the discrete scheme (2) approximates the NLS equation (1), (3) with an error of $O(\epsilon^2)$. To study more precisely how close equations (2) and (1) are and, in particular, the integrability properties of (2), in this paper we follow the procedure indicated in the first part of this introduction, obtaining the following results.

(1) Due to the structure of the vector field in (2), the generated ϵ-expansion contains only even powers. At $O(\epsilon^2)$, the dNLS (2) passes the DP test iff the ten coefficients satisfy the elegant quadratic constraint

$$(a_1 - 3a_3 - 2a_4 - 6a_5 - 5a_6 + 3a_7 - 5a_8 - 13a_9 - a_{10})\left(\sum_{j=1}^{10} a_j\right) = 0, \tag{9}$$

factorized into two linear constraints. If the first constraint $\sum_{j=1}^{10} a_j = 0$ is satisfied, we are in the C-integrability framework and the dNLS (2) approximates the linear Schrödinger (LS) equation with an error of $O(\epsilon^2)$, for time scales of $O(\epsilon^{-2})$. If, instead, the second constraint is satisfied:

$$a_1 - 3a_3 - 2a_4 - 6a_5 - 5a_6 + 3a_7 - 5a_8 - 13a_9 - a_{10} = 0, \tag{10}$$

we are in the S-integrability framework and the dNLS (2) approximates the NLS equation (1), (3) with an error of $O(\epsilon^2)$, for time scales of $O(\epsilon^{-2})$.

We remark that, among the ten single dNLS equations obtained choosing only one of the ten coefficients different from zero in (2), only the AL equation (4) satisfies the constraint (9) and passes the test at $O(\epsilon^2)$.

(2) At $O(\epsilon^4)$ we have the following two scenarios. In the C-integrability framework, the dNLS (2) approximates, with an error of $O(\epsilon^2)$, the linear Schrödinger equation for time scales of $O(\epsilon^{-4})$ iff the four linear constraints

$$\sum_{j=1}^{10} a_j = 0, \qquad a_1 + a_2 + a_6 + a_7 = 0, \qquad a_4 - a_5 + 2a_8 - 2a_9 = 0, \\ a_2 + 2(a_3 + 3a_5 + 3a_6 - a_7 + a_8 + 7a_9) = 0 \tag{11}$$

are satisfied by the coefficients. Since one of the real a_j can always be fixed rescaling the dependent variable ψ, equations (11) characterize a five-parameter family of discrete NLS equations (2) passing the test at such a high order.

In the S-integrability framework, the dNLS (2) approximates, with an error of $O(\epsilon^2)$, the NLS equation (1), (3) for time scales of $O(\epsilon^{-4})$ iff the coefficients satisfy, together with the linear constraint (10), the five quadratic constraints (53), (54)–(58). Since these five constraints do not contain the term $(a_2)^2$, they are trivially satisfied by the integrable AL equation (4), as it has to be. In general we do not expect a parametrization of such constraints in terms of elementary functions; however, we have been able to construct the following two explicit examples of dNLS equations:

$$\begin{aligned}&\mathrm{i}\psi_{n,t}+\epsilon^{-2}(\psi_{n+1}+\psi_{n-1}-2\psi_n)+a_6\big(-8|\psi_n|^2\psi_n+\tfrac{4}{3}|\psi_n|^2(\psi_{n+1}+\psi_{n-1})\\&\quad+4\psi_n^2(\bar\psi_{n+1}+\bar\psi_{n-1})-4\psi_n(\bar\psi_{n+1}\psi_{n-1}+\psi_{n+1}\bar\psi_{n-1})+\bar\psi_n(\psi_{n+1}^2+\psi_{n-1}^2)\\&\quad-2\bar\psi_n\psi_{n+1}\psi_{n-1}\big)=0,\end{aligned}\tag{12}$$

$$\begin{aligned}&\mathrm{i}\psi_{n,t}+\epsilon^{-2}(\psi_{n+1}+\psi_{n-1}-2\psi_n)+a_9\big(-48|\psi_n|^2\psi_n-8\psi_n(|\psi_{n+1}|^2\\&\quad+|\psi_{n-1}|^2)-8\psi_n(\bar\psi_{n+1}\psi_{n-1}+\psi_{n+1}\bar\psi_{n-1})+10\bar\psi_n(\psi_{n+1}^2+\psi_{n-1}^2)\\&\quad-4\bar\psi_n\psi_{n+1}\psi_{n-1}-7(|\psi_{n+1}|^2\psi_{n+1}+|\psi_{n-1}|^2\psi_{n-1})\\&\quad+(\psi_{n+1}^2\bar\psi_{n-1}+\psi_{n-1}^2\bar\psi_{n+1})+6(|\psi_{n+1}|^2\psi_{n-1}+|\psi_{n-1}|^2\psi_{n+1})\big)=0,\end{aligned}\tag{13}$$

satisfying such complicated quadratic constraints, corresponding to particular cases in which the associated five quadrics degenerate into hyperplanes.

These two distinguished models, passing the test at such a high order through the above degeneration mechanism, are obviously good candidates to be the S-integrable discretizations of NLS. A detailed study of their performances as numerical schemes for NLS, and of their possible integrability structure (Lax pair, etc), is postponed to a subsequent paper.

To obtain the above results, it is essential to use the well-known integrability properties of equation (1) (shared by all integrable systems; see, for instance [49–52]) that we summarize here, for completeness.

The NLS equation belongs to a hierarchy of infinitely many commuting flows:

$$u_{t_n}=K_n(u),\qquad n\in\mathbb{N},\tag{14}$$

i.e.

$$[K_n(u),K_m(u)]_L:=K_n'(u)[K_m(u)]-K_m'(u)[K_n(u)]=0,\qquad n,m\in\mathbb{N},\tag{15}$$

where

$$K_n'(u)[f]=\lim_{\epsilon\to0}\frac{\partial K_n}{\partial\epsilon}(u+\epsilon f)\tag{16}$$

is the usual Frechet derivative of $K_n(u)$ w.r.t u in the direction f. The commuting vector fields $\{K_n\}_{n\in\mathbb{N}}$ are arbitrary linear combinations, with constant coefficients, of the following basic symmetries $\{\sigma_n\}_{n\in\mathbb{N}}$, generated by the recursion relation:

$$\begin{aligned}&\sigma_{n+1}=\hat R\sigma_n,\qquad \sigma_0=-\mathrm{i}u,\qquad n\in\mathbb{N},\\&\hat Rf:=\mathrm{i}\big(f_x+2cu\partial_x^{-1}(u\bar f+\bar uf)\big),\end{aligned}\tag{17}$$

where $\hat{R}$ is the recursion operator of the NLS hierarchy [53]. The basic symmetries used in this paper are

$$\begin{aligned}
&\sigma_0 = -\mathrm{i}u, \qquad \sigma_2 = \mathrm{i}(u_{xx} + 2c|u|^2 u),\\
&\sigma_4 = -\mathrm{i}\big(u_{xxxx} + 2c(u^2\bar{u}_{xx} + 2u|u_x|^2 + 4|u|^2 u_{xx} + 3u_x^2\bar{u}) + 6c^2|u|^4 u\big),\\
&\sigma_6 = \mathrm{i}\big(u_{xxxxxx} + 2c(u^2\bar{u}_{xxxx} + 6|u|^2 u_{xxxx} + 4uu_x\bar{u}_{xxx} + 9u\bar{u}_x u_{xxx}\\
&\qquad + 15\bar{u}u_x u_{xxx} + 11u|u_{xx}|^2 + 10u_x^2\bar{u}_{xx}) + 10c^2(2u^2|u|^2\bar{u}_{xx} + 2\bar{u}u_{xx}^2\\
&\qquad + 5|u_x|^2 u_{xx} + 5|u|^4 u_{xx} + u^3\bar{u}_x^2 + 6u|u|^2|u_x|^2 + 7\bar{u}|u|^2 u_x^2) + 20c^3 u|u|^6\big),
\end{aligned} \tag{18}$$

and the NLS equation (1) corresponds to the flow $u_{t_2} = K_2(u) = \sigma_2(u)$.

Equivalently, the basic symmetries $\{\sigma_n\}_{n\in\mathbb{N}}$ are elements of the kernel of the 'linearized' nth flow operator $\hat{M}_n$, $n \in \mathbb{N}$, defined by

$$\hat{M}_n f := f_{t_n} - K_n'(u)[f]; \tag{19}$$

i.e.

$$\hat{M}_n \sigma_m = 0, \qquad n, m \in \mathbb{N}. \tag{20}$$

Due to (15), these linearized operators commute:

$$\hat{M}_n \hat{M}_m = \hat{M}_m \hat{M}_n, \qquad n, m \in \mathbb{N}. \tag{21}$$

The linearized operators used in this paper are

$$\begin{aligned}
&\hat{M}_2 f := f_{t_2} - \mathrm{i}\big(f_{xx} + 2c(u^2\bar{f} + 2|u|^2 f)\big),\\
&\hat{M}_4 f := f_{t_4} - \tfrac{\mathrm{i}}{12}\big[f_{xxxx} + 2c(\bar{u}(6u_x f_x + 4uf_{xx}) + u(2u_x\bar{f}_x + 2\bar{u}_x f_x\\
&\qquad + u\bar{f}_{xx}) + (6c|u|^2u^2 + 3u_x^2 + 4uu_{xx})\bar{f} + (9c|u|^4 + 2|u_x|^2 + 4\bar{u}u_{xx}\\
&\qquad + 2u\bar{u}_{xx})f)\big],\\
&\hat{M}_6 f := f_{t_6} - \tfrac{\mathrm{i}}{360}\big[f_{xxxxxx} + 2c(10cu^3(\bar{u}_x\bar{f}_x + \bar{u}\bar{f}_{xx}) + 5(2u_x^2\bar{f}_{xx}\\
&\qquad + 4\bar{u}u_{xx}f_{xx} + u_x(5\bar{u}_x f_{xx} + 5u_{xx}\bar{f}_x + 4\bar{u}_{xx}f_x + 3\bar{u}f_{xxx}) + (5\bar{u}_x u_{xx}\\
&\qquad + 3\bar{u}u_{xxx})f_x)) + u(70c\bar{u}^2u_x f_x + 11u_{xx}\bar{f}_{xx} + 11f_{xx}\bar{u}_{xx} + 9\bar{u}_x f_{xxx}\\
&\qquad + 4u_x\bar{f}_{xxx} + 9u_{xxx}\bar{f}_x + 6\bar{u}f_{xxxx}) + u^2(5c\bar{u}(6\bar{f}_x u_x + 6f_x\bar{u}_x + 5\bar{u}f_{xx})\\
&\qquad + \bar{f}_{xxxx}) + \bar{f}(30c^2|u|^4u^2 + 10cu^2(3|u_x|^2 + 5\bar{u}u_{xx}) + 10cu^3\bar{u}_{xx}\\
&\qquad + 5(2u_{xx}^2 + 3u_x u_{xxx}) + u(70c\bar{u}u_x^2 + 6u_{xxxx})) + f(40c^2|u|^6 + 35c\bar{u}^2u_x^2\\
&\qquad + 11|u_{xx}|^2 + 15cu^2(\bar{u}_x^2 + 2\bar{u}\bar{u}_{xx}) + 9\bar{u}_x u_{xxx} + 4u_x\bar{u}_{xxx} + 6\bar{u}u_{xxxx}\\
&\qquad + 2u(5c\bar{u}(6|u_x|^2 + 5\bar{u}u_{xx}) + \bar{u}_{xxxx})))\big].
\end{aligned} \tag{22}$$

At last, if $c = 0$, equations (1) and (17) lead to the LS equation

$$\mathrm{i}u_t + u_{xx} = 0 \tag{23}$$

and to its (trivial) symmetries $(-\mathrm{i}^{n+1}\partial_x^n u)$.

The paper is organized as follows. In section 2 we construct the multiscale expansion, in the small ϵ regime, of a generic solution of (2), establishing, in particular, that the leading term of such an expansion evolves w.r.t the infinitely many 'even' time variables $t_{2k} := \epsilon^{2(k-1)}t$, $k \in \mathbb{N}_+$, according to the even flows of the NLS hierarchy. In section 3, after summarizing the DP test and after showing how to use the main output of this test to construct infinitely many approximate symmetries of the original PΔE through novel and simple formulas, we apply the DP test to the PΔE (2), isolating the constraints on the coefficients a_j, $j = 1, \ldots, 10$, allowing one to pass the test at time scales of $O(\epsilon^{-2})$ and of

$O(\epsilon^{-4})$, in both scenarios of C- and S-integrability. In section 4 we summarize the results of the paper and we discuss the research perspectives opened by this work. In the appendix we display the long outputs of the DP test, obtained using the algebraic manipulation program of Mathematica.

2. Multiscale expansion in the small lattice spacing regime

If the lattice spacing ϵ is small: $0 < \epsilon \ll 1$, as a consequence of the invariance of (2) under the transformation $\psi_{n\pm 1} \to \psi_{n\mp 1}$ and of the well-known formula

$$f_{n\pm 1} = \sum_{k=0}^{\infty} \frac{(\pm 1)^k}{k!} \epsilon^k \partial_x^k f, \tag{24}$$

valid in the long-wave approximation, only even x-derivatives appear at all (even) orders in ϵ, implying that also the asymptotic expansion of ψ_n contains only even powers of ϵ. Consequently, to eliminate the secularities appearing at all even orders in ϵ, the coefficients of such an expansion must depend on infinitely many 'even' slow times [1]:

$$\vec{t} = (t_2, t_4, t_6, \ldots), \qquad t_{2k} := \epsilon^{2(k-1)} t, \qquad k \in \mathbb{N}_+, \tag{25}$$

implying that

$$\partial_t \to \partial_{t_2} + \epsilon^2 \partial_{t_4} + \epsilon^4 \partial_{t_6} + \cdots. \tag{26}$$

Therefore, we are led to the following ansatz for the asymptotic expansion of the 'generic' solution of (2):

$$\psi_n(t) = \sum_{k=0}^{\infty} \epsilon^{2k} u^{(2k+1)}(x, \vec{t}), \qquad u^{(1)}(x, \vec{t}) = u(x, \vec{t}). \tag{27}$$

Plugging (24), (26) and (27) into equation (2) and equating to zero the coefficients of all powers in ϵ, we obtain the following results.

At the leading $O(1)$, we obtain the NLS equation for the leading term $u^{(1)} = u$ w.r.t the first time $t_2 = t$:

$$\begin{aligned} &u_{t_2} = K_2(u), \\ &K_2(u) := \sigma_2(u) = \mathrm{i}(u_{xx} + 2c|u|^2 u), \qquad c = \sum_{j=1}^{10} a_j. \end{aligned} \tag{28}$$

As usual in perturbation theory, at the next relevant order ($O(\epsilon^2)$ in our case), the 'linearization' $\hat{M}_2 u^{(3)}$ of $(u_{t_2} - K_2(u))$ appears, together with the linear term $(u_{t_4} - (\mathrm{i}/12) u_{xxxx})$, coming from the linear part of (2), and with a nonlinear term G_5 coming from the nonlinear part of (2):

$$\hat{M}_2 u^{(3)} = -\left(u_{t_4} - \mathrm{i}\frac{2}{4!} u_{xxxx} \right) + G_5, \tag{29}$$

where

$$\begin{aligned} &G_5 = \mathrm{i}\left(s_1 u^2 \bar{u}_{xx} + s_2 |u|^2 u_{xx} + s_3 u |u_x|^2 + s_4 \bar{u} u_x^2 \right), \\ &s_1 = a_3 + a_4 + a_5 + a_8 + a_9 + a_{10}, \\ &s_2 = a_2 + a_4 + a_5 + 2(a_6 + a_7 + a_8 + a_9 + a_{10}), \\ &s_3 = 2(a_4 - a_5 + 2a_8 - 2a_9), \; s_4 = 2(a_6 - a_7 + a_8 + a_9 - a_{10}). \end{aligned} \tag{30}$$

Concentrating on the linear terms in the round bracket, we observe that $u_{t_4} \in \mathrm{Ker}\hat{M}_2$ and $(-(\mathrm{i}/12) u_{xxxx})$ is the linear part of the symmetry $(2/4!)\sigma_4(u) \in \mathrm{Ker}\hat{M}_2$. Therefore, adding and subtracting the symmetry $(2/4!)\sigma_4$, equation (29) is conveniently rearranged in the following way, isolating the resonant terms in the round bracket:

$$\hat{M}_2 u^{(3)} = -\left(u_{t_4} + \frac{2}{4!}\sigma_4(u)\right) + g_5, \tag{31}$$

where

$$g_5 := \mathrm{i}(c_1|u|^4 u + c_2 \bar{u} {u_x}^2 + c_3 u|u_x|^2 + c_4|u|^2 u_{xx} + c_5 u^2 \bar{u}_{xx}) \tag{32}$$

and

$$\begin{aligned}
c_1 &= -\tfrac{1}{2}c^2,\\
c_2 &= -\tfrac{1}{2}(a_1 + a_2 + a_3 + a_4 + a_5 - 3a_6 + 5a_7 - 3a_8 - 3a_9 + 5a_{10}),\\
c_3 &= -\tfrac{1}{3}(a_1 + a_2 + a_3 - 5a_4 + 7a_5 + a_6 + a_7 - 11a_8 + 13a_9 + a_{10}),\\
c_4 &= -\tfrac{1}{3}[2a_1 - a_2 + 2a_3 - a_4 - a_5 - 4(a_6 + a_7 + a_8 + a_9 + a_{10})],\\
c_5 &= -\tfrac{1}{6}[a_1 + a_2 + a_6 + a_7 - 5(a_3 + a_4 + a_5 + a_8 + a_9 + a_{10})].
\end{aligned} \tag{33}$$

To eliminate the secularity in the bracket, we are forced to choose

$$u_{t_4} = K_4(u) := -\frac{2}{4!}\sigma_4(u), \tag{34}$$

so (31) finally becomes the following secularity free equation for the first correction $u^{(3)}$:

$$\hat{M}_2 u^{(3)} = g_5. \tag{35}$$

This procedure iterates without essential differences at all orders. The terms $\hat{M}_2 u^{(3)}$ and $(u_{t_4} - K_4(u))$ in (31) generate, at $O(\epsilon^4)$, the terms $\hat{M}_2 u^{(5)}$ and $\hat{M}_4 u^{(3)}$ respectively, while the new linear term $(u_{t_6} - \mathrm{i}(2/6!)u_{xxxxxx})$ is rearranged again into the secular factor $(u_{t_6} - (2/6!)\sigma_6)$ that must be set to zero, to avoid secularities. Since, at $O(\epsilon^{2k})$, we produce the linear term $\left(u_{t_{2k}} - \mathrm{i}\frac{2}{(2k)!}\partial_x^{2k} u\right)$, one infers, in analogy with [1], that u evolves w.r.t the higher times according to the even flows of the NLS hierarchy as follows:

$$u_{t_{2k}} = K_{2k}(u) := (-1)^{k+1}\frac{2}{(2k)!}\sigma_{2k}(u), \qquad k \in \mathbb{N}_+, \tag{36}$$

and one is left with the following triangular set of equations [2, 3]:

$$\begin{aligned}
O(\epsilon^2):& \quad \hat{M}_2 u^{(3)} = g_5,\\
O(\epsilon^4):& \quad \hat{M}_2 u^{(5)} + \hat{M}_4 u^{(3)} = g_7,\\
O(\epsilon^6):& \quad \hat{M}_2 u^{(7)} + \hat{M}_4 u^{(5)} + \hat{M}_6 u^{(3)} = g_9,\\
&\vdots \qquad\qquad \vdots\\
O(\epsilon^{2k}):& \quad \hat{M}_2 u^{(2k+1)} + \hat{M}_4 u^{(2k-1)} + \cdots + \hat{M}_{2k} u^{(3)} = g_{2k+3},\\
&\vdots \qquad\qquad \vdots
\end{aligned} \tag{37}$$

where, for instance, the expression of g_7 is presented in formula (A.1) of the appendix. It remains to remark, following [2, 3], that the symmetries $\{\sigma_n\}$ and the expressions in (37), generated by the multiscale expansion, are differential polynomials satisfying the following properties: (i) they are linear combinations of products of the $u^{(k)}$'s and of their derivatives with respect to x: $\partial_x^j u^{(k)}$, $j \geqslant 0$, k odd; (ii) they possess the gauge symmetry of first kind. In addition, the differential polynomials appearing in the same equations exhibit the same 'order', in the following sense.

Definition 1. *If we define the order of the term* $\partial_x^j u^{(k)}$, $j \geqslant 0$, *as follows:*

$$\textit{order}\ (\partial_x^j u^{(k)}) = \textit{order}\ (\partial_x^j \overline{u^{(k)}}) = j + k, \tag{38}$$

then the order of the product of terms of this type is the sum of the orders of each term.

For example, order$(\partial_x^j u) = j + 1$ (since $u = u^{(1)}$) and order$(|u^{(k_1)}|^2 \partial_x^j u^{(k_2)}) = 2k_1 + j + k_2$. Therefore, we are naturally led to the definition of the following vector spaces.

Definition 2. $\mathcal{P}_n$ *is the vector space of all the differential polynomials satisfying properties (i) and (ii) above, of order n.* $\mathcal{P}_n(m)$ *is instead the subspace of* $\mathcal{P}_n$ *of all differential polynomials satisfying properties (i) and (ii) above and containing only terms* $(\partial_x^j u^{(k)})$, $(\partial_x^j \overline{u^{(k)}})$ *such that* $k \leqslant m$.

Is is easy to see that, for instance, $\sigma_n, K_n \in \mathcal{P}_{n+1}(1), g_5 \in \mathcal{P}_5(1)$ and $g_7 \in \mathcal{P}_7(3)$ (see (A.1)).

3. Applying the DP integrability test

Suppose we generate, from the model to be tested, an NLS-type multiscale expansion (as in our example); then we have the following scenarios. If such a model is S-integrable (C-integrable),

(1) the leading term u of the asymptotic expansion evolves, with respect to the slow times t_n, according to the NLS (LS) hierarchy [1];

(2) there exist elements $f_n^{(m)} \in \mathcal{P}_{n+m}$ such that the following equations hold [2, 3]:

$$\hat{M}_n u^{(m)} = f_n^{(m)} \in \mathcal{P}_{m+n}, \qquad m, n \in \mathbb{N}_+, \tag{39}$$

implying, due to (21), the compatibility conditions

$$\hat{M}_n f_m^{(j)} = \hat{M}_m f_n^{(j)}, \qquad m, n, j \in \mathbb{N}_+. \tag{40}$$

Therefore, equations (40) are necessary conditions to be satisfied, in cascade, for the model under investigation to be S- (C-)integrable; they are also sufficient to guaranty the asymptotic character of the expansion. If equations (40) are satisfied only up to a certain order, the model under investigation is not integrable, being nevertheless 'asymptotically integrable up to that order' [2, 3].

3.1. The DP test and approximate symmetries

Equations (39) and (40), the basic formulas of the DP test, have been derived in [2, 3] as a consequence of the existence of a Lax pair for the starting integrable model. It follows that if conditions (39) and (40) are satisfied up to a certain order, the equation under scrutiny admits an approximate Lax pair up to that order.

In this subsection we show how to derive conditions (39) and (40) from the existence of infinitely many symmetries of the starting integrable model. This derivation allows one to establish the important relations (to the best of our knowledge so far unknown) between the functions $f_n^{(m)} \in \mathcal{P}_{m+n}$ of the DP test and the symmetries of the starting model. We concentrate our attention on the case of difference equations, but our considerations have general validity.

Let $\psi_{n\,t_2} = \mathcal{K}_2(\psi_n)$ be an integrable model, say, the AL equation (4), and let $\psi_{n\,t_{2m}} = \mathcal{K}_{2m}(\psi_n), m > 2$, be one of its infinitely many higher order commuting flows (symmetries), reducing, in the continuous limit, to the higher commuting flow $u_{t_{2m}} = K_{2m}(u)$ of NLS.

On one hand, from equations (25) and (27), we have that

$$\psi_{n_{t_{2m}}} = u_{t_{2m}} + \epsilon^2(u_{t_{2(m+1)}} + u^{(3)}_{t_{2m}}) + \ldots = \sum_{k\geqslant 0}\epsilon^{2k}\left(\sum_{j=m}^{m+k} u^{(2(m+k-j)+1)}_{t_{2j}}\right), \tag{41}$$

where $u_{t_{2m}} = K_{2m}(u)$ (from (36)) and $u^{(2(m+k-j)+1)}_{t_{2j}} = K'_{2j}[u^{(2(m+k-j)+1)}] + f^{(2(m+k-j)+1)}_{2j}$, for some functions $f^{(2(m+k-j)+1)}_{2j}$ to be specified. On the other hand,

$$\mathcal{K}_{2m}(\psi_n) = K_{2m}(u) + \epsilon^2 K^{(2)}_{2m} + \ldots = K_{2m}(u) + \sum_{k\geqslant 1}\epsilon^{2k}K^{(2k)}_{2m}, \tag{42}$$

where $K^{(2k)}_{2m} \in \mathcal{P}_{2(m+k)+1}$. Equating equations (41) and (42), we infer that $f^{(m)}_n \in \mathcal{P}_{m+n}, m, n \in \mathbb{N}_+$ (the basic formula (39) of the DP test), and we also construct the asymptotic expansion of the generic higher order symmetry

$$\begin{aligned}\mathcal{K}_{2m}(\psi_n) &= K_{2m}(u) + \epsilon^2\big(K_{2(m+1)}(u) + K'_{2m}[u^{(3)}] + f^{(3)}_{2m}\big) + \ldots\\ &= \sum_{k\geqslant 0}\epsilon^{2k}\left(\sum_{j=m}^{m+k-1}\left(K'_{2j}[u^{(2(m+k-j)+1}] + f^{(2(m+k-j)+1)}_{2j}\right) + K_{2(m+k)}(u)\right)\end{aligned} \tag{43}$$

in terms of the NLS higher order symmetries, of their Frechet derivatives in the direction of the corrections $u^{(j)}$, $j > 1$, of the leading term u of expansion (27), and of the output functions $f^{(m)}_n \in \mathcal{P}_{m+n}$ of the DP test.

Therefore, if $f^{(2k+1)}_{2n} \in \mathcal{P}_{2(k+n)+1}$ exists, but $f^{(2k+3)}_{2n} \in \mathcal{P}_{2(k+n)+3}$ does not, $\forall n \in \mathbb{N}_+$, it follows that

(i) the solution $u^{(2k+1)}$ of (39) is uniformly bounded and expansion (27) is asymptotic up to $O(\epsilon^{2k})$; therefore, the PΔE under scrutiny approximates well its continuous limit, with an error of $O(\epsilon^2)$, for time scales up to $O(\epsilon^{-2k})$.

(ii) The PΔE possesses infinitely many 'approximate' generalized symmetries in the form (43) up to $O(\epsilon^{2k})$; therefore, it is integrable up to that order. We remark that, due to the Hamiltonian theory of integrable systems [49–52], it is also possible to associate with the PΔE infinitely many 'approximate' constants of motion in involution, a very useful information in a any numerical check.

3.2. C- and S-integrability at $O(\epsilon^2)$

In our example, the first of equations (37) is already in the form (39), with $g_5 = f^{(3)}_2 \in \mathcal{P}_5(1)$. Assuming now that $\hat{M}_4 u^{(3)} = f^{(3)}_4$, we arrive at the consistency

$$\hat{M}_4 f^{(3)}_2 = \hat{M}_2 f^{(3)}_4 \tag{44}$$

that must be viewed as an equation for the unknown $f^{(3)}_4$. Since $g_5 = f^{(3)}_2 \in \mathcal{P}_5(1)$, it follows that one must look for $f^{(3)}_4 \in \mathcal{P}_7(1)$. The calculation, plain but lengthy, has been performed using the algebraic manipulation program of Mathematica, and gives the following result.

Lemma 1. *Equation (44) admits a unique solution $f^{(3)}_4 \in \mathcal{P}_7(1)$ (presented in formula (A.4) of the appendix) iff the coefficients a_j's appearing in (2) satisfy the following quadratic constraint:*

$$(a_1 - 3a_3 - 2a_4 - 6a_5 - 5a_6 + 3a_7 - 5a_8 - 13a_9 - a_{10})\left(\sum_{j=1}^{10} a_j\right) = 0. \tag{45}$$

Once $f_4^{(3)}$ *is constructed,* $f_2^{(5)} \in \mathcal{P}_7(3)$ $(f_2^{(5)} = \hat{M}_2 u^{(5)})$ *is found from the second of equations (37):*

$$f_2^{(5)} = g_7 - f_4^{(3)} \tag{46}$$

and is presented in formula (A.7) of the appendix.

We first note the nice factorization of the quadratic constraint (45) into two linear constraints:

$$c = \sum_{j=1}^{10} a_j = 0, \tag{47}$$

$$a_1 - 3a_3 - 2a_4 - 6a_5 - 5a_6 + 3a_7 - 5a_8 - 13a_9 - a_{10} = 0. \tag{48}$$

Therefore, we have the following two different scenarios.

(1) If the first constraint (47) is satisfied by the coefficients a_j, the continuous limits of dNLS (2) are the LS equation. It follows that, in this case, equation (2) is 'asymptotically C-integrable' at $O(\epsilon^2)$ and one expects that, for generic initial data and at time scales of $O(\epsilon^{-2})$, the dynamics according to (2), (47) be well approximated by the dynamics according to the LS equation (23) with an error of $O(\epsilon^2)$.
In particular, the dNLS (2), (7) is 'asymptotically C-integrable' at $O(\epsilon^2)$ iff

$$a_1 + 4a_3 + 3a_6 = 0. \tag{49}$$

(2) If, instead, the second constraint (48) is satisfied by the coefficients a_j, the dNLS equation (2) is 'asymptotically S-integrable' at $O(\epsilon^2)$ and one expects that, for generic initial data and at time scales of $O(\epsilon^{-2})$, the dynamics according to the dNLS equation (2), (48) approximates well the dynamics according to the NLS equation (1), (3) with an error of $O(\epsilon^2)$.

In particular, (i) the dNLS (2), (6) is 'asymptotically S-integrable' at $O(\epsilon^2)$ iff the following additional constraint is satisfied:

$$a_3 + 2a_8 = 0; \tag{50}$$

(ii) the dNLS (2), (7) is 'asymptotically S-integrable' at $O(\epsilon^2)$ iff the following additional constraint is satisfied:

$$a_1 - 8a_3 - 9a_6 = 0, \tag{51}$$

while the dNLS (2), (8) is not 'asymptotically S-integrable' at $O(\epsilon^2)$ (therefore, it is not integrable).

In addition, since the dNLS equation (2) is the linear combination of ten different discretizations of NLS, it is immediate to check if some of these ten discretizations satisfy the constraint (48). Calling dNLS$_k$ the single discretization of NLS obtained choosing in (2) $a_j = a_k\delta_{jk}$, $j = 1, \ldots, 10$, it is straightforward to see (since the coefficient a_2 is the only one absent in (48)) that only the dNLS$_2$ equation (coinciding with the AL equation (4)) satisfies the constraint (48) (as it has to be, being an integrable system). All the other dNLS$_k$, $k \neq 2$ equations, including the dNLS$_1$ equation (5), do not satisfy the constraint (48); therefore, they are not 'asymptotically S-integrable' at $O(\epsilon^2)$ (consequently, they are not integrable) and, for generic initial data and at time scales of $O(\epsilon^{-2})$, their dynamics are expected to be quite different from that of NLS (1), (3), presumably exhibiting numerical evidence of nonintegrability and/or chaos.

We finally infer that the discretizations (2), (6) and (2), (7) satisfying respectively the constraints (50) and (51), the AL equation and any other dNLS equation (2) satisfying the constraint (48) are all close to NLS (once the free coefficients of each model are normalized to satisfy (3)) and are all close together at time scales of $O(\epsilon^{-2})$, in the sense mentioned in the introduction.

It is interesting to push the integrability test to the next order which we will present here. Due to the above factorization of the constraint (45), the test bifurcates and, in the next two subsections, we explore both cases. Before doing that, we observe that, given $f_2^{(3)} \in \mathcal{P}_5(1)$ and assuming that the constraint (45) be satisfied, the equation $\hat{M}_6 f_2^{(3)} = \hat{M}_2 f_6^{(3)}$ admits a unique solution $f_6^{(3)} = \hat{M}_6 u^{(3)} \in \mathcal{P}_9(1)$, presented in formula (A.10) of the appendix, and no additional constraint appears in this derivation, as predicted by the DP test.

3.3. *C-integrability at $O(\epsilon^4)$*

Let us assume that the constraint (47) be satisfied. For the construction of $f_4^{(5)} = \hat{M}_4 u^{(5)} \in \mathcal{P}_9(3)$ from the equation

$$\hat{M}_4 f_2^{(5)} = \hat{M}_2 f_4^{(5)} \tag{52}$$

we have the following result.

Lemma 2. *Equation (52) admits a unique solution $f_4^{(5)} \in \mathcal{P}_9(3)$, presented in formula (A.13) of the appendix, iff the coefficients a_j satisfy the four linear constraints (11), defining a six-parameter family (but one of these parameters can always be rescaled away) of dNLS equations (2) 'asymptotically C-integrable' at $O(\epsilon^4)$. Therefore, one expects that, for generic initial data and at time scales of $O(\epsilon^{-4})$, the dynamics according to (2), (11) well approximate the dynamics according to the LS equation (23) with an error of $O(\epsilon^2)$.*

For instance, the discretization (2), (7) satisfies the constraints (11) iff $a_6 = -a_3 = a_1$.

The six-parameter family of dNLS equations (2), (11) (or at least some particular case of it), being C-integrable at such a high order, is a natural candidate to be a C-integrable discrete system.

3.4. *S-integrability at $O(\epsilon^4)$*

Let us assume that the constraint (48) be satisfied. For the construction of a unique $f_4^{(5)} = \hat{M}_4 u^{(5)} \in \mathcal{P}_9(3)$ from equation (52), we have the following result.

Lemma 3. *If the constraint (48) is satisfied, equation (52) admits a unique solution $f_4^{(5)} \in \mathcal{P}_9(3)$, presented in formula (A.13) of the appendix, iff the following five quadratic constraints are satisfied:*

$$Q_j = 0, \quad j = 1, \ldots, 5, \tag{53}$$

where the Q_j's are the following quadratic forms in the nine variables $a_2, \ldots, a_{10}$:

$$\begin{aligned} Q_1 = {} & -4a_{10}^2 + a_{10}a_2 + 2a_{10}a_3 - a_2a_3 + 2a_3^2 - a_{10}a_4 - 2a_2a_4 + a_3a_4 \\ & + 3a_{10}a_5 - 2a_2a_5 - 3a_3a_5 - 8a_4a_5 - 8a_5^2 + 18a_{10}a_6 + 6a_3a_6 - 6a_{10}a_7 \\ & + 4a_2a_7 + 6a_3a_7 + 4a_4a_7 + 20a_5a_7 + 24a_6a_7 - 8a_7^2 + 12a_{10}a_8 - 3a_2a_8 \\ & + 6a_3a_8 + 3a_4a_8 - 9a_5a_8 - 6a_6a_8 + 18a_7a_8 + 20a_{10}a_9 - 3a_2a_9 - 2a_3a_9 \\ & - 13a_4a_9 - 25a_5a_9 - 6a_6a_9 + 50a_7a_9 - 24a_8a_9 - 24a_9^2, \end{aligned} \tag{54}$$

$$
\begin{aligned}
Q_2 = {} & 14a_{10}^2 + 6a_{10}a_2 + 44a_{10}a_3 + 4a_2a_3 + 26a_3^2 + 36a_{10}a_4 + 5a_2a_4 \\
& + 40a_3a_4 + 17a_4^2 + 72a_{10}a_5 + 7a_2a_5 + 88a_3a_5 + 68a_4a_5 + 75a_5^2 \\
& + 64a_{10}a_6 + 10a_2a_6 + 72a_3a_6 + 60a_4a_6 + 128a_5a_6 + 60a_6^2 - 24a_{10}a_7 \\
& - 2a_2a_7 - 24a_3a_7 - 16a_4a_7 - 44a_5a_7 - 32a_6a_7 + 4a_7^2 + 64a_{10}a_8 + 8a_2a_8 \\
& + 60a_3a_8 + 54a_4a_8 + 106a_5a_8 + 100a_6a_8 - 20a_7a_8 + 42a_8^2 + 168a_{10}a_9 \\
& + 28a_2a_9 + 220a_3a_9 + 170a_4a_9 + 382a_5a_9 + 332a_6a_9 - 108a_7a_9 \\
& + 284a_8a_9 + 466a_9^2,
\end{aligned} \tag{55}
$$

$$
\begin{aligned}
Q_3 = {} & 20a_{10}^2 + 15a_{10}a_2 + 38a_{10}a_3 - 5a_2a_3 - 22a_3^2 + 39a_{10}a_4 - a_2a_4 \\
& - 23a_3a_4 - a_4^2 + 63a_{10}a_5 - 11a_2a_5 - 83a_3a_5 - 52a_4a_5 - 75a_5^2 \\
& + 70a_{10}a_6 - 14a_2a_6 - 78a_3a_6 - 48a_4a_6 - 148a_5a_6 - 84a_6^2 - 18a_{10}a_7 \\
& + 10a_2a_7 + 42a_3a_7 + 32a_4a_7 + 76a_5a_7 + 88a_6a_7 - 20a_7^2 + 88a_{10}a_8 \\
& - 7a_2a_8 - 54a_3a_8 - 15a_4a_8 - 119a_5a_8 - 134a_6a_8 + 82a_7a_8 - 36a_8^2 \\
& + 72a_{10}a_9 - 59a_2a_9 - 302a_3a_9 - 211a_4a_9 - 539a_5a_9 - 526a_6a_9 + 234a_7a_9 \\
& - 472a_8a_9 - 788a_9^2,
\end{aligned} \tag{56}
$$

$$
\begin{aligned}
Q_4 = {} & -32a_{10}^2 - 24a_{10}a_2 - 56a_{10}a_3 + 6a_2a_3 + 36a_3^2 - 70a_{10}a_4 - a_2a_4 \\
& + 30a_3a_4 + a_4^2 - 114a_{10}a_5 + a_2a_5 + 78a_3a_5 + 20a_4a_5 + 27a_5^2 - 120a_{10}a_6 \\
& + 14a_2a_6 + 88a_3a_6 + 48a_4a_6 + 84a_5a_6 + 84a_6^2 + 24a_{10}a_7 - 14a_2a_7 - 64a_3a_7 \\
& - 52a_4a_7 - 80a_5a_7 - 112a_6a_7 + 28a_7^2 - 164a_{10}a_8 + 2a_2a_8 + 48a_3a_8 + 12a_4a_8 \\
& + 16a_5a_8 + 124a_6a_8 - 116a_7a_8 + 36a_8^2 - 220a_{10}a_9 + 22a_2a_9 + 208a_3a_9 + 96a_4a_9 \\
& + 196a_5a_9 + 292a_6a_9 - 204a_7a_9 + 176a_8a_9 + 300a_9^2,
\end{aligned} \tag{57}
$$

$$
\begin{aligned}
Q_5 = {} & 4a_{10}^2 + 3a_{10}a_2 - 2a_{10}a_3 - a_2a_3 - 14a_3^2 + 3a_{10}a_4 - a_2a_4 - 19a_3a_4 \\
& - 5a_4^2 - 5a_{10}a_5 - 3a_2a_5 - 47a_3a_5 - 36a_4a_5 - 39a_5^2 - 2a_{10}a_6 - 6a_2a_6 \\
& - 38a_3a_6 - 32a_4a_6 - 68a_5a_6 - 36a_6^2 + 6a_{10}a_7 + 2a_2a_7 + 18a_3a_7 + 16a_4a_7 \\
& + 28a_5a_7 + 24a_6a_7 - 4a_7^2 + 8a_{10}a_8 - 3a_2a_8 - 30a_3a_8 - 19a_4a_8 - 59a_5a_8 \\
& - 62a_6a_8 + 26a_7a_8 - 20a_8^2 - 40a_{10}a_9 - 23a_2a_9 - 150a_3a_9 - 119a_4a_9 - 255a_5a_9 \\
& - 230a_6a_9 + 82a_7a_9 - 216a_8a_9 - 356a_9^2.
\end{aligned} \tag{58}
$$

The five homogeneous quadratic constraints (53), (54)–(58) for nine unknowns, characterizing the intersection of five quadrics in the real projective space of dimension 8, define, in principle, a four-parameter family of solutions (but one of these parameters can always be rescaled away) whose parametrization does not appear to be expressible, in general, in terms of elementary functions. The corresponding dNLS equation (2) is asymptotically S-integrable at $O(\epsilon^4)$ and should well approximate the NLS equation for times up to $O(\epsilon^{-4})$.

We observe that, in all these quadratic constraints, a_2 is the only coefficient appearing always multiplied by other coefficients (the term $(a_2)^2$ is absent); therefore, the choice

$$a_j = a_2\delta_{j2}, \qquad j = 1, \ldots, 10, \tag{59}$$

corresponding to the AL equation (4) satisfies all constraints, as it has to be. Other less trivial explicit solutions of (48), (53), (54)–(58) can also be constructed, corresponding to the case in which all quadrics degenerate into pairs of hyperplanes. Here we display the following two examples:

$$\begin{array}{llll} a_1 = -4a_6, & a_2 = \frac{4a_6}{3}, & a_3 = 4a_6, & a_4 = 0, \\ a_5 = -4a_6, & a_7 = -a_6, & a_8 = a_9 = a_{10} = 0, & \end{array} \tag{60}$$

$$\begin{array}{llll} a_1 = -24a_9, & a_2 = a_3 = 0, & a_4 = a_5 = -8a_9, & \\ a_6 = 10a_9, & a_7 = -2a_9, & a_8 = -7a_9, & a_{10} = 6a_9, \end{array} \tag{61}$$

corresponding, respectively, to the dNLS equations (12) and (13) presented in the introduction, 'asymptotically S-integrable' at $O(\epsilon^4)$. Therefore, one expects that, for generic initial data and at time scales of $O(\epsilon^{-4})$, the dynamics according to equations (12) and (13) are good approximations of the dynamics according to the NLS equation (1), with an error of $O(\epsilon^2)$, at time scales of $O(\epsilon^{-4})$. Of course, these distinguished equations, passing the test at such a high order, are also good candidates to be S-integrable difference equations.

We finally observe that there is no choice of parameters for which the dNLS equations (2), (6), (50) and (2), (7), (51) satisfy the above constraints; therefore, these two models are not S-integrable at this order (they are not S-integrable at all) and do not approximate well NLS at time scales of $O(\epsilon^{-4})$.

4. Summary of the results and future perspectives

In this paper we have proposed an algorithmic procedure allowing one (i) to study the distance between an integrable PDE and any PΔE discretizing it, in the small lattice spacing ϵ regime; (ii) to test the (asymptotic) integrability properties of such a PΔE; and (iii) to construct infinitely many (approximate) symmetries and conserved quantities for it. This method should provide, in particular, useful and concrete information on how good is a numerical scheme used to integrate a given integrable PDE.

The procedure we have proposed, illustrated on the basic prototype example of the nonlinear Schrödinger equation (1) and of its discretization (2), consists of the following three steps: (i) the construction of the multiscale expansion of a generic long-wave solution of the dNLS (2) at all orders in ϵ, following [1]; (ii) the application, to such an expansion, of the DP integrability test [2, 3]; (iii) the use of the main output of such a test to construct infinitely many approximate symmetries of the dNLS equation (2), through novel formulas presented in this paper.

This approach allows one to study the distance between the integrable PDE and any PΔE discretizing it. Suppose, for instance, that the asymptotic expansion we construct reads $\psi = u + O(\epsilon^\alpha), \alpha > 0$, where ψ is a generic long-wave solution of the dNLS (2) and u is the corresponding solution of (1); then if the DP test is passed at $O(\epsilon^\beta)$, we conclude that (i) the dynamics according to the NLS equation (1) is well approximated (with an error of $O(\epsilon^\alpha)$) by the dynamics according to its discretization (2), for time scales of $O(t^{-\beta})$; (ii) the dNLS equation is asymptotically integrable up to that order, constructing its infinitely many approximate symmetries and constants of motion in involution. In contrast, if the DP test is not passed at that order, the dNLS equation is not integrable and one should expect, at the corresponding time scale, numerical evidence of nonintegrability and/or chaos.

We have carried the above procedure up to $O(\epsilon^4)$ and we have been able to isolate the constraints on the coefficients of the dNLS equation (2) allowing one to pass the test at that order, in both scenarios of S- and C-integrability.

Numerical experiments to test such theoretical findings are presently under investigation; preliminary results seem to confirm the theoretical predictions contained in this paper [54].

With the same methodology and goals, we are presently investigating families of discretizations of the Korteweg–de Vries and Burgers equations [55], other two basic integrable models of natural phenomena. Of course we also plan to investigate discretizations of integrable PDEs in which also the time variable is discretized.

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Appendix A.

In this appendix we display, for completeness, the long outputs of the DP test, obtained using the algebraic manipulation program of Mathematica.

The differential polynomial g_7 in (37) reads

$$
\begin{aligned}
g_7 = \mathrm{i}\big(&l_1 u|u|^6 + l_2|u|^4 u^{(3)} + l_3\bar{u}{u^{(3)}}^2 + l_4 u^2|u|^2\bar{u}^{(3)} + l_5 u|u^{(3)}|^2 \\
&+ l_6\bar{u}|u|^2 u_x^2 + l_7 u_x^2\bar{u}^{(3)} + l_8 u|u|^2|u_x|^2 + l_9|u_x|^2 u^{(3)} + l_{10}u^3\bar{u}_x^2 \\
&+ l_{11}\bar{u}u_x u_x^{(3)} + l_{12}u\bar{u}_x u_x^{(3)} + l_{13}uu_x\bar{u}_x^{(3)} + l_{14}|u|^4 u_{xx} + l_{15}\bar{u}u_{xx}u^{(3)} \\
&+ l_{16}uu_{xx}\bar{u}^{(3)} + l_{17}|u_x|^2 u_{xx} + l_{18}\bar{u}u_{xx}^2 + l_{19}u^2|u|^2\bar{u}_{xx} + l_{20}u\bar{u}_{xx}u^{(3)} \\
&+ l_{21}u_x^2\bar{u}_{xx} + l_{22}u|u_{xx}|^2 + l_{23}|u|^2 u_{xx}^{(3)} + l_{24}u^2\bar{u}_{xx}^{(3)} + l_{25}\bar{u}u_x u_{xxx} \\
&+ l_{26}u\bar{u}_x u_{xxx} + l_{27}uu_x\bar{u}_{xxx} + l_{28}|u|^2 u_{xxxx} + l_{29}u^2\bar{u}_{xxxx}\big),
\end{aligned} \tag{A.1}
$$

where

$$
\begin{aligned}
&l_1 = -\tfrac{1}{18}c^3, \qquad l_2 = -\tfrac{3}{2}c^2, \qquad l_3 = 2c, \qquad l_4 = -c^2, \qquad l_5 = 4c, \\
&l_6 = 7l_{10}, \qquad l_7 = \tfrac{1}{2}l_{11}, \qquad l_8 = 6l_{10}, \\
&l_9 = -\tfrac{1}{3}(a_1 + a_2 + a_3 - 5a_4 + 7a_5 + a_6 + a_7 - 11a_8 + 13a_9 + a_{10}), \\
&l_{10} = -\tfrac{1}{36}c^2, \qquad l_{11} = -(a_1 + a_2 + a_3 + a_4 + a_5 - 3a_6 + 5a_7 - 3(a_8 + a_9) + 5a_{10}), \\
&l_{12} = l_{13}, \qquad l_{13} = -\tfrac{1}{3}(a_1 + a_2 + a_3 - 5a_4 + 7a_5 + a_6 + a_7 - 11a_8 + 13a_9 + a_{10}), \\
&l_{14} = 5l_{10}, \qquad l_{15} = l_{16}, \\
&l_{16} = -\tfrac{1}{3}(2a_1 - a_2 + 2a_3 - a_4 - a_5 - 4(a_6 + a_7 + a_8 + a_9 + a_{10}), \\
&l_{17} = -\tfrac{1}{36}(5(a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_{10}) - 67a_8 + 77a_9), \\
&l_{18} = -\tfrac{1}{18}(a_1 + a_2 + a_3 + a_4 + a_5 - 8(a_6 + a_7 + a_8 + a_9 + a_{10})), \\
&l_{19} = 2l_{10}, \qquad l_{20} = -\tfrac{1}{3}(a_1 + a_2 + a_6 + a_7 - 5(a_3 + a_4 + a_5 + a_8 + a_9 + a_{10})), \\
&l_{21} = -\tfrac{1}{18}(a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 - 17(a_8 + a_9) + 19a_{10}),
\end{aligned} \tag{A.2}
$$

$$
\begin{aligned}
&l_{22} = -\tfrac{1}{180}(11(a_1 + a_2 + a_3) - 79(a_4 + a_5) + 11(a_6 + a_7) - 169(a_8 + a_9 + a_{10})), \\
&l_{23} = -\tfrac{1}{3}(2a_1 - a_2 + 2a_3 - a_4 - a_5 - 4(a_6 + a_7 + a_8 + a_9 + a_{10})), \qquad l_{24} = \tfrac{1}{2}l_{20}, \\
&l_{25} = -\tfrac{1}{12}(a_1 + a_2 + a_3 + a_4 + a_5 + 9a_7 - 7(a_6 + a_8 + a_9) + 9a_{10}), \\
&l_{26} = -\tfrac{1}{60}(3(a_1 + a_2 + a_3 + a_{10}) - 17a_4 + 23a_5 + 3a_6 + 3a_7 - 37a_8 + 43a_9), \\
&l_{27} = -\tfrac{1}{45}(a_1 + a_2 + a_3 - 14a_4 + 16a_5 + a_6 + a_7 - 29a_8 + 31a_9 + a_{10}),
\end{aligned}
$$

$$l_{28} = -\tfrac{1}{60}(2a_1 - 3a_2 + 2a_3 - 3a_4 - 3a_5 - 8(a_6 + a_7 + a_8 + a_9 + a_{10})),$$
$$l_{29} = -\tfrac{1}{180}(a_1 + a_2 + a_6 + a_7 - 14(a_3 + a_4 + a_5 + a_8 + a_9 + a_{10})). \quad \text{(A.3)}$$

The solution $f_4^{(3)} \in \mathcal{P}_7(1)$ of $\hat{M}_4 f_2^{(3)} = \hat{M}_2 f_4^{(3)}$, where $f_2^{(3)} = g_5$ is given in (32), (33), exists unique and reads

$$\begin{aligned} f_4^{(3)} = \mathrm{i}\big(&\alpha_1 u|u|^6 + \alpha_2 u_{xx}|u|^4 + \alpha_3 \bar{u}_{xx} u^2 |u|^2 + \alpha_4 u_x^2 |u|^2 \bar{u} + \alpha_5 |u_x|^2 |u|^2 u \\ &+ \alpha_6 \bar{u}_x^2 u^3 + \alpha_7 u_{xxxx}|u|^2 + \alpha_8 \bar{u}_{xxxx} u^2 + \alpha_9 u_{xxx} u_x \bar{u} + \alpha_{10} \bar{u}_{xxx} u_x u + \alpha_{11} u_{xxx} \bar{u}_x u \\ &+ \alpha_{12} u_{xx}^2 \bar{u} + \alpha_{13} |u_{xx}|^2 u + \alpha_{14} u_{xx} |u_x|^2 + \alpha_{15} \bar{u}_{xx} {u_x}^2\big), \end{aligned} \quad \text{(A.4)}$$

where

$$\begin{aligned} &\alpha_1 = \frac{c^2}{3}(2c_2 - c_3 + c_4 + 3c_5), \qquad \alpha_2 = \frac{c}{6}(4c_2 - 2c_3 + 6c_4 + 5c_5), \\ &\alpha_3 = \frac{c}{12}(2c_2 - c_3 + 3c_4 + 10c_5), \qquad \alpha_4 = \frac{c}{24}(40c_2 - 11c_3 + 25c_4 + 20c_5), \\ &\alpha_5 = \frac{c}{12}(8c_2 + 5c_3 + 7c_4 + 16c_5), \qquad \alpha_6 = \frac{c}{24}(4c_2 + c_3 + c_4 + 8c_5), \\ &\alpha_7 = \frac{c_4}{6}, \qquad \alpha_8 = \frac{c_5}{12}, \qquad \alpha_9 = \frac{1}{12}(4c_2 + 3c_4), \qquad \alpha_{10} = \frac{1}{12}(c_3 + 2c_5), \\ &\alpha_{11} = \frac{1}{12}(2c_3 + c_4), \qquad \alpha_{12} = \frac{1}{12}(3c_2 + 2c_4), \qquad \alpha_{13} = \frac{1}{12}(c_3 + c_4 + 4c_5), \\ &\alpha_{14} = \frac{1}{12}(2c_2 + 5c_3 + c_4), \qquad \alpha_{15} = \frac{1}{12}(c_2 + c_3 + 3c_5), \end{aligned} \quad \text{(A.5)}$$

iff the following constraint is satisfied:

$$2c_1 - c(2c_2 - c_3 + c4 + 4c_5) = 0 \quad \text{(A.6)}$$

on the coefficients c_j's defined in (33). This constraint is equivalent to (45).

$f_2^{(5)} = g_7 - f_4^{(3)}$ consequently reads, from $(A.1)$ and $(A.4)$,

$$\begin{aligned} f_2^{(5)} = \mathrm{i}\big(&d_1 u|u|^6 + d_2 |u|^4 u^{(3)} + d_3 \bar{u} {u^{(3)}}^2 + d_4 u^2 |u|^2 \bar{u}^{(3)} + d_5 u |u^{(3)}|^2 \\ &+ d_6 \bar{u}|u|^2 u_x^2 + d_7 u_x^2 \bar{u}^{(3)} + d_8 u|u|^2 |u_x|^2 + d_9 |u_x|^2 u^{(3)} + d_{10} u^3 \bar{u}_x^2 + d_{11} \bar{u} u_x u_x^{(3)} \\ &+ d_{12} u \bar{u}_x u_x^{(3)} + d_{13} u u_x \bar{u}_x^{(3)} + d_{14} |u|^4 u_{xx} + d_{15} \bar{u} u_{xx} u^{(3)} + d_{16} u u_{xx} \bar{u}^{(3)} \\ &+ d_{17} |u_x|^2 u_{xx} + d_{18} \bar{u} u_{xx}^2 + d_{19} u^2 |u|^2 \bar{u}_{xx} + d_{20} u \bar{u}_{xx} u^{(3)} + d_{21} u_x^2 \bar{u}_{xx} + d_{22} u |u_{xx}|^2 \\ &+ d_{23} |u|^2 u_{xx}^{(3)} + d_{24} u^2 \bar{u}_{xx}^{(3)} + d_{25} \bar{u} u_x u_{xxx} + d_{26} u \bar{u}_x u_{xxx} + d_{27} u u_x \bar{u}_{xxx} \\ &+ d_{28} |u|^2 u_{xxxx} + d_{29} u^2 \bar{u}_{xxxx}\big), \end{aligned} \quad \text{(A.7)}$$

where

$$\begin{aligned} d_1 &= \frac{c^2}{9}(5a_1 + 2a_2 - 4a_3 - a_4 - 13a_5 - 13a_6 + 11a_7 - 10a_8 - 34a_9 + 2a_{10}), \\ d_2 &= -\frac{3}{2}c^2, \qquad d_3 = 2c, \qquad d_4 = -c^2, \quad d_5 = 4c, \\ d_6 &= \frac{c}{72}(95a_1 + 20a_2 + 35a_3 + 26a_4 - 106a_5 - 295a_6 + 185a_7 - 223a_8 - 487a_9 \\ &\quad + 125a_{10}), \\ d_7 &= -\frac{1}{2}(a_1 + a_2 + a_3 + a_4 + a_5 - 3a_6 + 5a_7 - 3(a_8 + a_9) + 5a_{10}), \end{aligned} \quad \text{(A.8)}$$

$$
\begin{aligned}
d_8 &= \frac{c}{12}(11a_1 + 4a_2 - 5a_3 - 22a_4 - 2a_5 - 19a_6 + 13a_7 - 55a_8 - 15a_9 - 3a_{10}),\\
d_9 &= d_{13}, \qquad d_{10} = \frac{c}{72}(11a_1 + 8a_2 - 13a_3 - 22a_4 - 10a_5 - 19a_6 + 29a_7 - 55a_8\\
&\quad - 31a_9 + 5a_{10}), \qquad d_{11} = 2d_7, \qquad d_{12} = d_{13},\\
d_{13} &= -\frac{1}{3}(a_1 + a_2 + a_3 - 5a_4 + 7a_5 + a_6 + a_7 - 11a_8 + 13a_9 + a_{10}),\\
d_{14} &= \frac{c}{18}(16a_1 - 2a_2 + a_3 - 5a_4 - 29a_5 - 44a_6 + 4a_7 - 35a_8 - 83a_9 - 11a_{10}),\\
d_{15} &= d_{16}, \qquad d_{16} = -\frac{1}{3}(2a_1 - a_2 + 2a_3 - a_4 - a_5 - 4(a_6 + a_7 + a_8 + a_9 + a_{10})),\\
d_{17} &= \frac{1}{36}(5a_1 + 2a_2 + 5a_3 - 28a_4 + 32a_5 - 13a_6 + 11a_7 - a_8 - 25a_9 + 11a_{10}),\\
d_{18} &= \frac{1}{72}(13a_1 + a_2 + 13a_3 + a_4 + a_5 - 11a_6 + 61a_7 - 11(a_8 + a_9) + 61a_{10}),\\
d_{19} &= \frac{c}{36}(11a_1 + 2a_2 - 19a_3 - 22a_4 - 34a_5 - 19a_6 + 5a_7 - 37a_8 - 61a_9 - 25a_{10}),\\
d_{20} &= -\frac{1}{3}(a_1 + a_2 + a_6 + a_7 - 5(a_3 + a_4 + a_5 + a_8 + a_9 + a_{10})), \qquad\qquad (A.9)\\
d_{21} &= \frac{1}{36}(2a_1 + 2a_2 - 7a_3 - 13a_4 - a_5 - 4a_6 + 8a_7 + 11a_8 + 35a_9 - 37a_{10}),\\
d_{22} &= \frac{1}{180}(14a_1 - a_2 - 46a_3 - a_4 + 59a_5 - 16a_6 - 16a_7 + 44a_8 + 164a_9\\
&\quad + 104a_{10}),\\
d_{23} &= -\frac{1}{3}(2a_1 - a_2 + 2a_3 - a_4 - a_5 - 4(a_6 + a_7 + a_8 + a_9 + a_{10})), \quad d_{24} = \frac{1}{2}d_{20},\\
d_{25} &= \frac{1}{4}(a_1 + a_3 - a_6 - a_7 - a_8 - a_9 - a_{10}),\\
d_{26} &= \frac{1}{180}(11a_1 - 4a_2 + 11a_3 - 4a_4 - 4a_5 - 19(a_6 + a_7 + a_8 + a_9 + a_{10})),\\
d_{27} &= \frac{1}{30}(a_1 + a_2 - 4a_3 + a_4 - 9a_5 + a_6 + a_7 + 6a_8 - 14a_9 - 4a_{10}),\\
d_{28} &= \frac{1}{180}(14a_1 - a_2 + 14a_3 - a_4 - a_5 - 16(a_6 + a_7 + a_8 + a_9 + a_{10})),\\
d_{29} &= \frac{c}{120}.
\end{aligned}
$$

The unique solution $f_6^{(3)} = \hat{M}_6 u^{(3)} \in \mathcal{P}_9(1)$ of equation $\hat{M}_6 f_2^{(3)} = \hat{M}_2 f_6^{(3)}$ reads

$$
\begin{aligned}
f_6^{(3)} = \mathrm{i}\big(&\beta_1|u|^8u + \beta_2u_{xx}|u|^6 + \beta_3\bar{u}_{xx}|u|^4u^2 + \beta_4u_x^2|u|^4\bar{u} + \beta_5|u_x|^2|u|^4u\\
&+ \beta_6\bar{u}_x^2u^3|u|^2 + \beta_7u_{xxxx}|u|^4 + \beta_8u_{xxx}u_x|u|^2\bar{u} + \beta_9u_{xxx}\bar{u}_x|u|^2u + \beta_{10}u_{xx}^2|u|^2\bar{u}\\
&+ \beta_{11}|u_{xx}|^2|u|^2u + \beta_{12}\bar{u}_{xxx}u_x|u|^2u + \beta_{13}\bar{u}_{xxx}\bar{u}_xu^3 + \beta_{14}u_{xx}u_x^2\bar{u}^2\\
&+ \beta_{15}u_{xx}|u_x|^2|u|^2 + \beta_{16}u_{xx}\bar{u}_x^2u^2 + \beta_{17}u_x^3\bar{u}_x\bar{u} + \beta_{18}|u_x|^4u + \beta_{19}(\bar{u}_{xx})^2u^3\\
&+ \beta_{20}\bar{u}_{xx}|u_x|^2u^2 + \beta_{21}\bar{u}_{xx}u_x^2|u|^2 + \beta_{22}u_{xxxxxx}|u|^2 + \beta_{23}\bar{u}_{xxxxxx}u^2
\end{aligned}
$$

$$+ \beta_{24}u_{xxxxx}|u|^2 + \beta_{25}\bar{u}_{xxxxx}uu_x + \beta_{26}u_{xxxx}|u_x|^2 + \beta_{27}u_{xxxx}u_{xx}\bar{u}$$
$$+ \beta_{28}u_{xxxx}\bar{u}_{xx}u + \beta_{29}\bar{u}_{xxxx}u_x^2 + \beta_{30}\bar{u}_{xxxx}u_{xx}u + \beta_{31}u_{xxx}^2\bar{u} + \beta_{32}u_{xxx}u_{xx}\bar{u}_x$$
$$+ \beta_{33}u_{xxx}\bar{u}_{xx}u_x + \beta_{34}|u_{xxx}|^2u + \beta_{35}|u_{xx}|^2u_{xx} + \beta_{36}u_{xxxxx}u_x\bar{u}$$
$$+ \beta_{37}u_{xx}\bar{u}_{xxx}u_x + \beta_{38}\bar{u}_{xxxx}|u|^2u^2\big), \quad \text{(A.10)}$$

where

$$\beta_1 = \frac{c^3}{48}(6c_2 - 3c_3 + 3c_4 + 8c_5), \qquad \beta_2 = \frac{c^2}{36}(10c_2 - 5c_3 + 10c_4 + 12c_5),$$
$$\beta_3 = \frac{c^2}{36}(4c_2 - 2c_3 + 4c_4 + 9c_5), \qquad \beta_4 = \frac{c^2}{18}(13c_2 - 5c_3 + 9c_4 + 10c_5),$$
$$\beta_5 = \frac{c^2}{36}(18c_2 - 3c_3 + 14c_4 + 28c_5), \qquad \beta_6 = \frac{c^2}{36}(5c_2 - c_3 + 3c_4 + 8c_5),$$
$$\beta_7 = \frac{c}{180}(6c_2 - 3c_3 + 15c_4 + 7c_5), \qquad \beta_8 = \frac{c}{360}(108c_2 - 29c_3 + 126c_4 + 56c_5), \quad \text{(A.11)}$$
$$\beta_9 = \frac{c}{360}(36c_2 + 7c_3 + 66c_4 + 52c_5), \qquad \beta_{10} = \frac{c}{720}(152c_2 - 41c_3 + 169c_4 + 84c_5),$$
$$\beta_{11} = \frac{c}{360}(44c_2 - 7c_3 + 79c_4 + 118c_5), \qquad \beta_{12} = \frac{c}{360}(16c_2 + 7c_3 + 26c_4 + 62c_5),$$
$$\beta_{13} = \frac{c}{360}(8c_2 + c_3 + 6c_4 + 26c_5), \qquad \beta_{14} = \frac{c}{72}(33c_2 - 7c_3 + 21c_4 + 14c_5),$$
$$\beta_{15} = \frac{c}{360}(278c_2 + 11c_3 + 276c_4 + 236c_5), \qquad \beta_{16} = \frac{c}{360}(39c_2 + 8c_3 + 33c_4 + 68c_5),$$

$$\beta_{17} = \frac{c}{72}(22c_2 - c_3 + 13c_4 + 16c_5), \qquad \beta_{18} = \frac{c}{720}(158c_2 + 31c_3 + 101c_4 + 176c_5),$$
$$\beta_{19} = \frac{c}{720}(12c_2 - c_3 + 9c_4 + 44c_5), \qquad \beta_{20} = \frac{c}{120}(16c_2 + 7c_3 + 12c_4 + 42c_5),$$
$$\beta_{21} = \frac{c}{360}(114c_2 - 12c_3 + 103c_4 + 158c_5), \qquad \beta_{22} = \frac{c_4}{120}, \qquad \beta_{23} = \frac{c_5}{360},$$
$$\beta_{24} = \frac{1}{240}(2c_3 + 3c_4), \qquad \beta_{25} = \frac{1}{360}(c_3 + 4c_5), \qquad \beta_{26} = \frac{1}{720}(18c_2 + 21c_3 + 25c_4),$$
$$\beta_{27} = \frac{1}{360}(15c_2 + 13c_4), \qquad \beta_{28} = \frac{1}{720}(9c_3 + 11c_4 + 12c_5),$$
$$\beta_{29} = \frac{1}{360}(c_2 + 2c_3 + 10c_5), \qquad \beta_{30} = \frac{1}{360}(2c_3 + c_4 + 11c_5),$$
$$\beta_{31} = \frac{1}{144}(4c_2 + 3c_4), \qquad \beta_{32} = \frac{1}{720}(50c_2 + 35c_3 + 34c_4),$$
$$\beta_{33} = \frac{1}{360}(11c_2 + 17c_3 + 10c_4 + 15c_5), \qquad \beta_{34} = \frac{1}{720}(11c_3 + 4c_4 + 18c_5),$$
$$\beta_{35} = \frac{1}{720}(20c_2 + 25c_3 + 11c_4 + 20c_5),$$
$$\beta_{36} = \frac{1}{240}(4c_2 + 5c_4), \qquad \beta_{37} = \frac{1}{720}(8c_2 + 31c_3 + 4c_4 + 50c_5),$$
$$\beta_{38} = \frac{c}{360}(2c_2 - c_3 + 5c_4 + 14c_5), \quad \text{(A.12)}$$

and no additional constraint on the coefficients c_j's appears. The unique solution $f_4^{(5)} = \hat{M}_4 u^{(5)} \in \mathcal{P}_9(3)$ of equation $\hat{M}_4 f_2^{(5)} = \hat{M}_2 f_4^{(5)}$ reads

$$
\begin{aligned}
f_4^{(5)} = \mathrm{i}\big(&\delta_1 u|u|^8 + \delta_2 u_{xx}|u|^6 + \delta_3 \bar{u}_{xx} u^2|u|^4 + \delta_4 u_x^2 \bar{u}|u|^4 + \delta_5|u_x|^2 u|u|^4 \\
&+ \delta_6 \bar{u}_x^2|u|^2 u^3 + \delta_7 u_{xxxx}|u|^4 + \delta_8 u_{xxx} u_x|u|^2\bar{u} + \delta_9 u_{xxx}\bar{u}_x|u|^2 u + \delta_{10} u_{xx}^2|u|^2\bar{u} \\
&+ \delta_{11}|u_{xx}|^2|u|^2 u + \delta_{12}\bar{u}_{xxx} u_x|u|^2 u + \delta_{13}\bar{u}_{xxx}\bar{u}_x u^3 + \delta_{14} u_{xx} u_x^2\bar{u}^2 \\
&+ \delta_{15} u_{xx}|u_x|^2|u|^2 + \delta_{16} u_{xx}\bar{u}_x^2 u^2 + \delta_{17} u_x^3\bar{u}_x\bar{u} + \delta_{18}|u_x|^4 u + \delta_{19}\bar{u}_{xx}^2 u^3 \\
&+ \delta_{20}\bar{u}_{xx}|u_x|^2 u^2 + \delta_{21}\bar{u}_{xx} u_x^2|u|^2 + \delta_{22} u_{xxxxxx}|u|^2 + \delta_{23}\bar{u}_{xxxxxx} u^2 \\
&+ \delta_{24} u_{xxxxx}\bar{u}_x u + \delta_{25}\bar{u}_{xxxxx} u_x u + \delta_{26} u_{xxxx}|u_x|^2 + \delta_{27} u_{xxxx} u_{xx}\bar{u} \\
&+ \delta_{28} u_{xxxx}\bar{u}_{xx} u + \delta_{29}\bar{u}_{xxxx} u_x^2 + \delta_{30}\bar{u}_{xxxx} u_{xx} u + \delta_{31} u_{xxx}^2\bar{u} \\
&+ \delta_{32} u_{xxx} u_{xx}\bar{u}_x + \delta_{33} u_{xxx}\bar{u}_{xx} u_x + \delta_{34}|u_{xxx}|^2 u + \delta_{35} u_{xx}|u_{xx}|^2 \\
&+ \delta_{36} u_{xxxxx} u_x\bar{u} + \delta_{37} u_{xx}\bar{u}_{xxx} u_x + \delta_{38} u^2|u|^2\bar{u}_{xxxx} + \gamma_1 u^{(3)}_{xxxx}|u|^2 + \gamma_2 u^{(3)}_{xxx} u_x\bar{u} \\
&+ \gamma_3 u^{(3)}_{xxx}\bar{u}_x u + \gamma_4 u^{(3)}_{xx} u_{xx}\bar{u} + \gamma_5 u^{(3)}_{xx}|u_x|^2 + \gamma_6 u^{(3)}_{xx} u\bar{u}_{xx} + \gamma_7 u^{(3)}_x u_{xxx}\bar{u} \\
&+ \gamma_8 u^{(3)}_x u_{xx}\bar{u}_x + \gamma_9 u^{(3)}_x u_x\bar{u}_{xx} + \gamma_{10} u^{(3)}_x u\bar{u}_{xxx} + \gamma_{11} u^{(3)} u_{xxxx}\bar{u} + \gamma_{12} u^{(3)} u_{xxx}\bar{u}_x \\
&+ \gamma_{13} u^{(3)}|u_{xx}|^2 + \gamma_{14} u^{(3)} u_x\bar{u}_{xxx} + \gamma_{15} u^{(3)} u\bar{u}_{xxxx} + \gamma_{16}\bar{u}^{(3)}_{xxxx} u^2 + \gamma_{17}\bar{u}^{(3)}_{xxx} u_x u \\
&+ \gamma_{18}\bar{u}^{(3)}_{xx} u_{xx} u + \gamma_{19}\bar{u}^{(3)}_{xx} u_x^2 + \gamma_{20}\bar{u}^{(3)}_x u_{xxx} u + \gamma_{21}\bar{u}^{(3)}_x u_{xx} u_x + \gamma_{22}\bar{u}^{(3)} u_{xxxx} u \\
&+ \gamma_{23}\bar{u}^{(3)} u_{xxx} u_x + \gamma_{24}\bar{u}^{(3)} u_{xx}^2 + \gamma_{25} u^{(3)}_{xx}|u|^4 + \gamma_{26} u^{(3)}_x u_x|u|^2\bar{u} + \gamma_{27} u^{(3)}_x\bar{u}_x|u|^2 u \\
&+ \gamma_{28} u^{(3)} u_{xx}|u|^2\bar{u} + \gamma_{29} u^{(3)} u_x^2\bar{u}^2 + \gamma_{30} u^{(3)}|u_x|^2|u|^2 + \gamma_{31} u^{(3)}\bar{u}_x^2 u^2 \\
&+ \gamma_{32} u^{(3)}\bar{u}_{xx}|u|^2 u + \gamma_{33}\bar{u}^{(3)}_{xx}|u|^2 u^2 + \gamma_{34}\bar{u}^{(3)}_x u_x|u|^2 u + \gamma_{35}\bar{u}^{(3)}_x\bar{u}_x u^3 \\
&+ \gamma_{36}\bar{u}^{(3)} u_{xx}|u|^2 u + \gamma_{37}\bar{u}^{(3)} u_x^2|u|^2 + \gamma_{38}\bar{u}^{(3)}|u_x|^2 u^2 + \gamma_{39}\bar{u}^{(3)}\bar{u}_{xx} u^3 + \gamma_{40} u^{(3)}|u|^6 \\
&+ \gamma_{41}\bar{u}^{(3)}|u|^4 u^2 + \sigma_1 u^{(3)}_{xx} u^{(3)}\bar{u} + \sigma_2 {u^{(3)}_x}^2\bar{u} + \sigma_3 u^{(3)}_x u^{(3)}\bar{u}_x + \sigma_4 {u^{(3)}}^2\bar{u}_{xx} \\
&+ \sigma_5 u^{(3)}_{xx}\bar{u}^{(3)} u + \sigma_6|u^{(3)}_x|^2 u + \sigma_7 u^{(3)}_x\bar{u}^{(3)} u_x + \sigma_8 u^{(3)}\bar{u}^{(3)}_{xx} u + \sigma_9 u^{(3)}\bar{u}^{(3)}_x u_x \\
&+ \sigma_{10}|u^{(3)}|^2 u_{xx} + \sigma_{11}(u^{(3)})^2|u|^2\bar{u} + \sigma_{12}|\bar{u}^{(3)}|^2|u|^2 u + \sigma_{13}(\bar{u}^{(3)})^2 u^3\big),
\end{aligned}
\tag{A.13}
$$

where

$$
\begin{aligned}
\delta_1 = \frac{c}{576}\big(&336d_1 + 4(6c_2 - 3c_3 + 2c_4 + 10c_5)d_2 + 6c_5(2c_2 - c_3 + 2c_4 + 4c_5)d_3 \\
&+ 4(6c_2 - 3c_3 + 3c_4 + 2c_5)d_4 + \big(-4c_4^2 + 2c_2c_5 - c_3c_5 + c_4c_5 - 2c_5^2\big)d_5 \\
&+ c(2(14c_2 - 5c_3 + 9c_4 + 28c_5)d_{11} - 2(10c_2 - 3c_3 + 7c_4 + 8c_5)d_{12} - 4(6c_2 \\
&- 3c_3 + c_4 + 16c_5)d_{13} + 16d_{14} + 2(18c_2 - 7c_3 + 11c_4 + 30c_5)d_{15} - 2(6c_2 \\
&- 3c_3 + 3c_4 + 14c_5)d_{16} - 48d_{19} - 4(6c_2 - 3c_3 + 3c_4 + 8c_5)d_{20} + 4(-4c_2 + c_3 \\
&+ 2c_4 + 15c_5)d_{23} + 4(-6c_2 + 5c_3 + c_4 - 6c_5)d_{24} + 32d_6 - 16d_8 + 4(2c_2 - c_3 \\
&+ c_4 + 4c_5)d_9) + 8c^2(-2d_{17} + 4d_{18} + 2d_{22} + 7d_{25} - 7d_{26} - 5d_{28} + 16d_{29})\big),
\end{aligned}
$$

$$
\begin{aligned}
\delta_2 = \frac{1}{144}\big(&96d_1 + 10c_4d_2 + 4c_4c_5d_3 - 6c_5d_4 - 2c_4^2d_5 + c(2(4c_2 - c_3 + 6c_4 \\
&+ 8c_5)d_{11} - 2(2c_2 + 5c_4)d_{12} - 2(6c_2 - 3c_3 + c_4 + 14c_5)d_{13} + 104d_{14} + (18c_2 \\
&- 7c_3 + 11c_4 + 24c_5)d_{15} + 3(-2c_2 + c_3 - c_4 - 4c_5)d_{16} - 24d_{19} + (34c_2 - 19c_3
\end{aligned}
$$

$$
\begin{aligned}
&+31c_4+80c_5)d_{23}+2(-6c_2+5c_3-5c_4-10c_5)d_{24}+16d_6-8d_8+2(2c_2\\
&-c_3+c_4+4c_5)d_9)+4c^2(-2d_{17}+4d_{18}+2d_{22}+13d_{25}-13d_{26}+d_{28}+4d_{29})\big),\\
\delta_3 &= \frac{1}{48}(12d_1+2c_5d_2+2c_4d_4+c(2c_5d_{11}-2c_5d_{12}+8d_{14}+24d_{19}+(2c_2\\
&-c_3+c_4+4c_5)d_{20}+2(2c_2-c_3+c_4+5c_5)d_{23}+5(2c_2-c_3+c_4)d_{24})\\
&+8c^2(d_{25}-d_{26}+d_{28}-2d_{29}))),\\
\delta_4 &= \frac{1}{144}\big(216d_1+2(3c_2+6c_4-2c_5)d_2+2\big(2c_3c_4+2c_2c_5+c_3c_5+c_4c_5\\
&-2c_5^2\big)d_3-24c_5d_4-\big(6c_4^2-3c_3c_5+c_4c_5+2c_5^2\big)d_5+c(-24d_{10}+(52c_2-19c_3\\
&+9c_4+54c_5)d_{11}+2(-7c_2+3c_3-8c_4+9c_5)d_{12}+2(-10c_2+4c_3-5c_4\\
&-22c_5)d_{13}+28d_{14}+2(33c_2-4c_3+5c_4+16c_5)d_{15}+2(-22c_2+12c_3-18c_4\\
&-19c_5)d_{16}-102d_{19}+(4c_2-10c_3-6c_4+7c_5)d_{20}+(154c_2-58c_3+121c_4\\
&+199c_5)d_{23}+3(-22c_2+18c_3-3c_4-22c_5)d_{24}+132d_6+(10c_2-5c_3+5c_4\\
&+4c_5)d_7-16d_8+2(6c_2-3c_3+3c_4+13c_5)d_9)+2c^2(-42d_{17}+118d_{18}-8d_{21}\\
&+17d_{22}+54d_{25}-37d_{26}+7d_{27}+193d_{28}+160d_{29})\big),\\
\delta_5 &= \frac{1}{48}\big(48d_1+2(3c_3-2c_4+4c_5)d_2+2c_4c_5d_3+2(-c_3-6c_4+4c_5)d_4\\
&+2\big(c_4^2-c_5^2\big)d_5+c(-8d_{10}+4(c_2+4c_5)d_{11}+(10c_2-7c_3+11c_4+8c_5)d_{12}\\
&+(6c_2-3c_3+3c_4+4c_5)d_{13}+24d_{14}-4(3c_2-c_3+c_4+c_5)d_{15}+4(4c_2\\
&-2c_3+3c_4+3c_5)d_{16}+16d_{19}+4c_5d_{20}+2(28c_2-15c_3+6c_4+42c_5)d_{23}\\
&+4(-4c_2+c_3+5c_4-17c_5)d_{24}+8d_6+32d_8+(-2c_2+c_3-c_4-4c_5)d_9)\\
&+4c^2(2d_{17}+4d_{18}-6d_{22}+3d_{25}+d_{26}-4d_{27}-13d_{28}-4d_{29})\big),\\
\delta_6 &= \frac{1}{48}\big(12d_1+2c_5^2d_3+2(c_2-c_4)d_4-c_4c_5d_5+c(20d_{10}+2(2c_2\\
&-c_3+4c_5)d_{12}-4d_{14}+2(c_2+2c_5)d_{15}-2(c_4+c_5)d_{16}+8d_{19}+(8c_2\\
&-2c_3+7c_4+8c_5)d_{23}+2(10c_2-5c_3+11c_4+5c_5)d_{24}+4d_6+4d_8)\\
&+2c^2(-2d_{17}+12d_{18}-2d_{22}+d_{25}+3d_{26}-4d_{27}+9d_{28}+12d_{29})\big),\\
\delta_7 &= \frac{1}{24}(4d_{14}+2c_4d_{23}-c_5d_{24}+2c(d_{25}-d_{26}+9d_{28}-2d_{29})),\\
\delta_8 &= \frac{1}{24}(c_4d_{11}-c_5d_{13}+8d_{14}+2c_2d_{15}+(6c_2+7c_4)d_{23}-2c_5d_{24}\\
&+8d_6+2c(-2d_{17}+8d_{18}+12d_{25}-2d_{26}-d_{27}+27d_{28}-4d_{29})),\\
\delta_9 &= \frac{1}{24}(c_4d_{12}+4d_{14}+2c_5d_{15}-2c_4d_{16}+(3c_3+4c_4)d_{23}-c_3d_{24}+4d_8\\
&+2c(4d_{18}-2d_{22}+d_{25}+9d_{26}-d_{27}+18d_{28})),
\end{aligned}
\tag{A.14}
$$

$$
\begin{aligned}
\delta_{10} &= \frac{1}{24}(2c_4d_{11}-2c_5d_{13}+8d_{14}+3c_4d_{15}-c_5d_{16}+2c_2d_{23}+c_4d_{23}+6d_6\\
&+2c(11d_{18}-d_{22}+2d_{25}-2d_{27}+6d_{28})),\\
\delta_{11} &= \frac{1}{24}(2c_5d_{11}-2c_4d_{13}+4d_{14}+c_5d_{15}+c_4d_{16}+12d_{19}+c_4d_{20}+(4c_3+c_4\\
&+6c_5)d_{23}+(-c_4+2c_5)d_{24}+2d_8+4c(d_{18}+4d_{22}+d_{25}-d_{26}+13d_{28}-2d_{29})),
\end{aligned}
$$

$$\delta_{12} = \frac{1}{24}(c_5d_{11} + c_4d_{13} + 6d_{19} + c_3d_{23} + 10c_5d_{23} + c_3d_{24} - 4c_4d_{24} + 2d_8 + 2c(d_{25} + d_{26} + 7d_{27} + 8d_{28} + 8d_{29})),$$

$$\delta_{13} = \frac{1}{24}(4d_{10} + c_5d_{12} + 2d_{19} + 2c_2d_{24} + 2c_4d_{24} + 2c(d_{25} + d_{26} + d_{27} + 8d_{29})),$$

$$\delta_{14} = \frac{1}{48}(4(3c_2 + c_4)d_{11} + 8d_{14} + 2(2c_2 - c_4)d_{15} - 2c_5d_{16} + (12c_2 + 7c_4)d_{23} + 2c_5d_{24} + 24d_6 - 6c_5d_7 + 2c(-4d_{17} + 20d_{18} - 6d_{21} - 2d_{22} + 37d_{25} - 2d_{26} + 37d_{28} + 4d_{29})),$$

$$\delta_{15} = \frac{1}{24}((5c_3 + c_4)d_{11} + (2c_2 + c_4)d_{12} + (-3c_3 + 2c_5)d_{13} + 8d_{14} + (c_3 + 4c_5)d_{15} + (c_3 - 4c_4)d_{16} + (8c_2 + 11c_3 + 6c_4 - 4c_5)d_{23} + 4(-c_3 + c_4 - 2c_5)d_{24} + 8d_6 - 4c_4d_7 + 20d_8 + c_4d_9 + 4c(4d_{17} + 8d_{18} + 11d_{25} + 5d_{26} - 7d_{27} + 34d_{28} - 8d_{29})),$$

$$\delta_{16} = \frac{1}{48}(12d_{10} - 2c_5d_{11} + 2(2c_3 + c_4 + 3c_5)d_{12} - 2(2c_2 + c_4)d_{13} + 4d_{14} + 2c_2d_{16} + 6d_{19} - 3c_5d_{20} - (2c_3 + 7c_5)d_{23} + (4c_2 - 2c_3 + c_4 + 4c_5)d_{24} + 8d_8 + 2c_5d_9 + 2c(2d_{17} + 6d_{18} + 2d_{21} + d_{22} - 3d_{25} + 26d_{26} - 11d_{27} - 14d_{28} + 4d_{29})),$$

$$\delta_{17} = \frac{1}{24}((c_2 + 3c_3)d_{11} - 2c_3d_{13} + 2(c_2 + c_3 - c_5)d_{23} + 2c_4d_{24} + 4d_6 + (c_3 - 2c_4)d_7 + 6d_8 + c_2d_9 + 4c(d_{17} + 2d_{21} + 7d_{25} - 4d_{27} + 6d_{28})),$$

$$\delta_{18} = \frac{1}{24}(6d_{10} - 2c_5d_{11} + (c_2 + 3c_3 + 6c_5)d_{12} - (3c_2 - c_3 + 2c_4)d_{13} + 6d_{19} - 3c_5d_{20} - (2c_3 + 7c_5)d_{23} + (2c_2 - 6c_3 + c_4)d_{24} + 2d_6 + c_2d_7 + 8d_8 + c_3d_9 + 2c_5d_9 + 2c(4d_{17} + 6d_{18} + 2d_{21} - 3d_{22} - 3d_{25} + 20d_{26} - 5d_{27} - 14d_{28} - 20d_{29})),$$

$$\delta_{19} = \frac{1}{24}(2d_{10} + 2d_{19} + c_5d_{20} + (2c_2 + c_4)d_{24} + 2c(d_{18} + d_{22} + 6d_{29})),$$

$$\delta_{20} = \frac{1}{24}(24d_{10} + (c_3 + 2c_5)d_{12} + 2c_2d_{13} + c_4d_{13} + 12d_{19} + (c_3 + 3c_5)d_{20} - 3c_5d_{23} - (14c_2 - 3c_3 + 7c_4)d_{24} + 4d_8 + c_5d_9 + 2c(2d_{17} - 6d_{18} + 2d_{21} + 7d_{22} - 3d_{25} + 6d_{26} + 9d_{27} - 36d_{29})),$$

$$\delta_{21} = \frac{1}{24}((c_3 + 6c_5)d_{11} + (c_3 - 3c_4)d_{13} + 18d_{19} + c_2d_{20} + (c_2 + 6c_3 + 10c_5)d_{23} - 4(c_3 + c_4)d_{24} + 4d_6 + c_4d_7 + 4d_8 + 4c(4d_{21} + d_{22} + 5d_{25} - 2d_{26} + 3d_{27} + 20d_{28} - 16d_{29})),$$

$$\delta_{22} = \frac{d_{28}}{6}, \qquad \delta_{23} = \frac{d_{29}}{12}, \qquad \delta_{24} = \frac{1}{12}(2d_{26} + d_{28}), \qquad \delta_{25} = \frac{1}{12}(d_{27} + 2d_{29}),$$

$$\delta_{26} = \frac{1}{12}(2d_{17} + d_{25} + 3d_{26} + d_{28}), \qquad \delta_{27} = \frac{1}{12}(4d_{18} + 3d_{25} + 2d_{28}), \tag{A.15}$$

$$\delta_{28} = \frac{1}{12}(2d_{22} + d_{26} + d_{28}), \qquad \delta_{29} = \frac{1}{12}(d_{21} + d_{27} + 3d_{29}),$$
$$\delta_{30} = \frac{1}{12}(d_{22} + d_{27} + 4d_{29}), \qquad \delta_{31} = \frac{1}{12}(3d_{18} + 2d_{25}),$$
$$\delta_{32} = \frac{1}{12}(5d_{17} + 2d_{18} + d_{25} + 2d_{26}), \quad \delta_{33} = \frac{1}{12}(d_{17} + 4d_{21} + 3d_{22} + d_{25} + d_{26}),$$
$$\delta_{34} = \frac{1}{12}(d_{22} + d_{26} + 2d_{27}), \qquad \delta_{35} = \frac{1}{12}(d_{17} + d_{18} + 3d_{21} + 2d_{22}),$$
$$\delta_{36} = \frac{1}{12}(2d_{25} + 3d_{28}), \qquad \delta_{37} = \frac{1}{12}(d_{17} + 2d_{21} + d_{22} + 5d_{27}),$$
$$\delta_{38} = \frac{1}{24}(2d_{19} + c_5 d_{23} + c_4 d_{24} + 4c(d_{28} + 3d_{29})),$$
$$\gamma_1 = \frac{d_{23}}{6}, \qquad \gamma_2 = \frac{1}{12}(2d_{11} + 3d_{23}), \qquad \gamma_3 = \frac{1}{12}(2d_{12} + d_{23}),$$
$$\gamma_4 = \frac{1}{12}(3d_{11} + 2d_{15} + 2d_{23}), \qquad \gamma_5 = \frac{1}{12}(d_{11} + 3d_{12} + d_{23} + 2d_9),$$
$$\gamma_6 = \frac{1}{12}(d_{12} + 2d_{20} + d_{23}), \qquad \gamma_7 = \frac{1}{12}(2d_{11} + 3d_{15}),$$
$$\gamma_8 = \frac{1}{12}(d_{11} + 2d_{12} + d_{15} + 3d_9), \qquad \gamma_9 = \frac{1}{12}(d_{11} + d_{12} + 3d_{20} + d_9),$$
$$\gamma_{10} = \frac{1}{12}(d_{12} + d_{20}), \qquad \gamma_{11} = \frac{d_{15}}{6}, \qquad \gamma_{12} = \frac{1}{12}(d_{15} + 2d_9),$$
$$\gamma_{13} = \frac{1}{12}(d_{15} + 2d_{20} + d_9), \qquad \gamma_{14} = \frac{1}{12}(d_{20} + d_9), \qquad \gamma_{15} = \frac{d_{20}}{12},$$
$$\gamma_{16} = \frac{d_{24}}{12}, \qquad \gamma_{17} = \frac{1}{12}(d_{13} + 2d_{24}), \qquad \gamma_{18} = \frac{1}{12}(d_{13} + d_{16} + 4d_{24}),$$
$$\gamma_{19} = \frac{1}{12}(d_{13} + 3d_{24} + d_7), \qquad \gamma_{20} = \frac{1}{12}(2d_{13} + d_{16}), \qquad \gamma_{21} = \frac{1}{12}(5d_{13} + d_{16} + 2d_7),$$
$$\gamma_{22} = \frac{d_{16}}{6}, \qquad \gamma_{23} = \frac{1}{12}(3d_{16} + 4d_7), \qquad \gamma_{24} = \frac{1}{12}(2d_{16} + 3d_7),$$
$$\gamma_{25} = \frac{1}{12}(2d_2 + c(d_{11} - d_{12} + 9d_{23} - 2d_{24})),$$
$$\gamma_{26} = \frac{1}{12}\big(6d_2 + 2c_4 d_3 - c_5 d_5 + c(11d_{11} - d_{12} - d_{13} + 2d_{15} - 2d_{20} + 10d_{23} - 8d_{24})\big),$$
$$\gamma_{27} = \frac{1}{12}(2d_2 + 2c_5 d_3 - c_4 d_5 + c(d_{11} + 7d_{12} + d_{13} + 2d_{15} - 2d_{16} + 8d_{23})),$$
$$\gamma_{28} = \frac{1}{24}(8d_2 + 6c_4 d_3 - c_5 d_5 + 2c(2d_{11} - 2d_{13} + 11d_{15} - d_{16} - d20 + 2d_{23})),$$
$$\gamma_{29} = \frac{1}{24}(6d_2 + 2(c_2 + c_3 - 2c_5)d_3 - (c_3 - c_4)d_5 + 2c(4d_{11} - 2d_{13} + 4d_{15} - 2d_{20} + 2d_{23} + d_7 + d_9)),$$
$$\gamma_{30} = \frac{1}{24}(8d_2 + 2(c_3 + 4c_5)d_3 + (c_3 - 4c_4)d_5 + 4c(d_{11} + d_{12} + d_{13} + 4d_{15} - 2d_{16} + 2d_{20} + 6d_{23} - 4d_{24} + 4d_9)),$$
$$\gamma_{31} = \frac{1}{24}(2d_2 + c_2 d_5 + 2c(2d_{12} + 2d_{20} + 2d_{24} + d_7 + d_9)),$$

$$\gamma_{32} = \frac{1}{24}(4d_2 + 2c_5d_3 + c_4d_5 + 2c(d_{15} + d_{16} + 7d_{20} + 2d_{23} + 2d_{24})),$$

$$\gamma_{33} = \frac{1}{2}(d_4 + 2c(d_{23} + 3d_{24})),$$

$$\gamma_{34} = \frac{1}{12}(3d_4 + c(d_{11} + d_{12} + 7d_{13} + 4d_{23} + 4d_{24})),$$

$$\gamma_{35} = \frac{1}{12}(d_4 + c(d_{11} + d_{12} + d_{13} + 4d_{24})),$$

$$\gamma_{36} = \frac{1}{24}(12d_4 + c_4d_5 + 2c(2d_{11} - 2d_{13} + 3d_{15} + 7d_{16} - d_{20} + 6d_{23} - 2d_{24})),$$

$$\gamma_{37} = \frac{1}{24}(18d_4 + c_2d_5 + 4c(3d_{11} - 2d_{12} + d_{13} + d_{16} + 7d_{23} - 8d_{24} + 4d_7),$$

$$\gamma_{38} = \frac{1}{24}(18d_4 + c_3d_5 + 4c(d_7 + d_9 - 3d_{11} + 4d_{12} + d_{13} + 2d_{16} - 3d_{23} - 10d_{24})),$$

$$\gamma_{39} = \frac{1}{24}(2d_4 + c_5d_5 + 2c(d_{15} + d_{16} + d_{20} + 2d_{24})),$$

$$\gamma_{40} = \frac{c}{144}(80d_2 + 2(6c_2 - 3c_3 + c_4 + 12c_5)d_3 + 16d_4 + (10c_2 - 5c_3 + 5c_4 + 8c_5)d_5 + 8c(3d_{11} - d_{12} - 2d_{13} + 3d_{15} - d_{16} - 2d_{20} + 2d_{23} - 2d_{24})),$$

$$\gamma_{41} = \frac{c}{48}(8d_2 + 24d_4 + (2c_2 - c_3 + c_4 + 4c_5)d_5 + 8c(d_{11} - d_{12} + d_{23} - 2d_{24})),$$

$$\sigma_1 = \frac{d_3}{3}, \sigma_2 = \frac{d_3}{4}, \sigma_3 = \frac{d_3}{6}, \sigma_4 = \frac{d_3}{12}, \sigma_5 = \sigma_{10} = \frac{d_5}{6}, \sigma_6 = \frac{d_5}{12}, \sigma_7 = \frac{d_5}{4},$$

$$\sigma_8 = \sigma_9 = \frac{d_5}{12}, \sigma_{11} = \frac{c}{12}(7d_3 + d_5), \sigma_{12} = \frac{c}{6}(d_3 + 4d_5), \sigma_{13} = \frac{c}{12}(d_3 + d_5), \qquad \text{(A.16)}$$

iff the following three sets of constraints are satisfied:

$$d_{15} - d_{16} = d_5 - 2d_3 = c_4d_3 - 2cd_{23} = -d_{12} + d_{13} - d_{15} + d_{20} + d_{23} - 2d_{24}$$
$$= -d_9 + d_{11} + d_{13} - 2d_7 = (2c_2 - c_3 + c_4)d_3 - 2(d_{11} - d_{13} + d_{16} - d_{20} + 2d_{24})c \qquad \text{(A.17)}$$
$$= (c_2 - c_3 + c_4 + c_5)d_3 + 2(d_{13} - d_{16} - d_{24} - d_7)c = 0,$$

$$(2c_2 - c_3 + c_4 + 2c_5)d_3 - 2d_4 + 4d_{24}c$$
$$= -4d_2 + (6c_2 - 3c_3 + 3c_4 + 8c_5)d_3 + 8d_{24}c = 0, \qquad \text{(A.18)}$$

$$-4d_{10} + 2c_5d_{13} - 2c_5d_{16} + 2d_{19} + c_5d_{20} + c_5d_{23} + (2c_2 - c_4 - 4c_5)d_{24}$$
$$+ 2c(2d_{18} - d_{22} - d_{25} + 2d_{26} - d_{27} - 2d_{28} + 4d_{29}) = 0,$$
$$-4d_{14} - 2c_4d_{11} + 2(c_2 - c_4 + c_5)d_{16} + (4c_2 + 3c_4)d_{23} + 2c_5d_{24} + 2c(-2d_{17}$$
$$+ 4d_{18} + 2d_{22} + d_{25} - d_{26} + 13d_{28} + 4d_{29}) = 0,$$
$$-2d_8 - (c_4 - 2c_5)d_7 - 2(c_4 - c_5)d_{11} + (2c_2 + c_3 + 2c_4 + c_5)d_{13} - (c_3 + 3c_4)d_{16}$$
$$+ 6d_{19} + (-c_3 + 2c_4 + c_5)d_{20} + (5c_2 - c_3 + 3c_4)d_{23} - (4c_2 + c_3 + 6c_4)d_{24}$$
$$+ 2c(2d_{17} - 2d_{18} - 4d_{21} - 7d_{22} - d_{25} + 2d_{26} + 11d_{27} + 10d_{28} - 20d_{29}) = 0,$$

$$
\begin{aligned}
&-4d_6+(2c_2+2c_3-c_4-4c_5)d_7-12d_{10}+(c_3+2c_5)d_{11}-(c_2-4c_4+9c_5)d_{13}\\
&\quad -4d_{14}+(3c_2-5c_5)d_{16}+12d_{19}-(2c_2+c_3+2c_4-4c_5)d_{20}+(c_2+4c_3+c_4\\
&\quad -c_5)d_{23}+(6c_2-2c_3+3c_4-12c_5)d_{24}+(-8d_{17}+20d_{18}+4d_{21}-14d_{22}+10d_{25}\\
&\quad +4d_{26}+10d_{27}+44d_{28}-16d_{29})=0,\\
&-24d_1+6\left(c_5^2-c_4c_5\right)d_3+2(6c_2-c_4)d_4+(2(2c_2+2c_3-c_4+4c_5)d_7-4d_8\\
&\quad -52d_{10}+2(-4c_2+c_3-c_4+2c_5)d_{11}+2(c_2+c_3-6c_4+6c_5)d_{13}-16d_{14}+2(5c_2\\
&\quad -c_3+c_4-c_5)d_{16}+62d_{19}+(-4c_2+2c_3-12c_4+5c_5)d_{20}+(2c_2+12c_3+4c_4\\
&\quad +13c_5)d_{23}+(6c_2-12c_3+5c_4)d_{24})c+2c^2(-16d_{17}+50d_{18}+4d_{21}-13d_{22}+d_{25}\\
&\quad +20d_{26}+3d_{27}+52d_{28}+4d_{29})=0.
\end{aligned}
\tag{A.19}
$$

The first set of constraints (A.17) is automatically satisfied by the parametrizations (33) and (A.8), while the second set of two constraints (A.18) is satisfied by the parametrizations (33) and (A.8) and by the $O(\epsilon^2)$ constraint (45). The remaining five constraints (A.19) are equivalent to the five quadratic constraints (53), (54)–(58) in the S-integrability scenario in which (48) holds, and to the linear constraints (11) in the C-integrability scenario in which $c=0$ (the last constraint is automatically satisfied by the condition $c=0$ and, in the remaining constraints, the quadrics degenerate into the hyperplanes described by equations (11)).

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